Algebra Preliminary Exam — September 2005

Instructions: Do as many problems as you can. Single complete solutions are better than several partial solutions. Correct answers to four problems is a pass. Do not reprove major theorems unless asked to do so, but when you use such theorems say so. In writing down partial solutions try to indicate the gaps as clearly as possible, so that we can see what you do and don't know.

1. For any group G we define $\Omega(G)$ to be the image of the group homomorphism $\rho : G \to \mathrm{Aut}(G)$ where ρ maps $g \in G$ to the conjugation automorphism $x \to gxg^{-1}$. Starting with a group G_0, we define $G_1 = \Omega(G_0)$ and $G_{i+1} = \Omega(G_i)$ for all $i \geq 1$. If G_0 is of order p^e for a prime p and integer $e \geq 2$, prove that G_{e-1} is the trivial group.

2. Let $\mathbb{F}_2$ be the field with 2 elements.

(a) What is the order of $GL_3(\mathbb{F}_2)$?

(b) Use the fact that $GL_3(\mathbb{F}_2)$ is a simple group (which you should not prove) to find the number of elements of order 7 in $GL_3(\mathbb{F}_2)$.

3. Let G be a finite abelian group. Let $f : \mathbb{Z}^m \to G$ be a surjection of abelian groups. We may think of f as a homomorphism of $\mathbb{Z}$-modules. Let K be the kernel of f.

(a) Prove that K is isomorphic to $\mathbb{Z}^m$.

(b) We can therefore write the inclusion map $K \to \mathbb{Z}^m$ as $\mathbb{Z}^m \to \mathbb{Z}^m$ and represent it by an $m \times m$ integer matrix A. Prove that $|\det A| = |G|$.

4. Let $R = C([0,1])$ be the ring of all continuous real-valued functions on the closed interval $[0,1]$, and for each $c \in [0,1]$, denote by M_c the set of all functions $f \in R$ such that $f(c) = 0$.

(a) Prove that $g \in R$ is a unit if and only if $g(c) \neq 0$ for all $c \in [0,1]$.

(b) Prove that for each $c \in [0,1]$, M_c is a maximal ideal of R.

(c) Prove that if M is a maximal ideal of R, then $M = M_c$ for some $c \in [0,1]$. (Hint: compactness of $[0,1]$ may be relevant.)

5. Let R and S be commutative rings, and $f : R \to S$ a ring homomorphism.

(a) Show that if I is a prime ideal of S, then

$$f^{-1}(I) = \{r \in R : f(r) \in I\}$$

is a prime ideal of R.

(b) Let N be the set of nilpotent elements of R:

$$N = \{r \in R : r^m = 0 \text{ for some } m \geq 1\}.$$

N is called the *nilradical* of R. Prove that it is an ideal which is contained in every prime ideal.

(c) Part (a) lets us define a function

$$\begin{aligned} f^* : \{\text{prime ideals of } S\} &\longrightarrow \{\text{prime ideals of } R\}. \\ I &\longmapsto f^{-1}(I) \end{aligned}$$

Let N be the nilradical of R. Show that if $S = R/N$ and $f : R \to R/N$ is the quotient map, then f^* is a bijection.

6. Let F be a finite field of characteristic p. Let A be an $n \times n$ matrix over F. Suppose that A^p is the identity matrix. Show that for every polynomial $f(x)$, the characteristic polynomial of the matrix $f(A)$ is equal to $(t-c)^n$ for some c.

7. Consider the polynomial $f(x) = x^{10} + x^5 + 1 \in \mathbb{Q}[x]$ with splitting field K over $\mathbb{Q}$.

(a) Determine whether $f(x)$ is irreducible over $\mathbb{Q}$ and find $[K : \mathbb{Q}]$.

(b) Determine the structure of the Galois group $\mathbf{Gal}(K/\mathbb{Q})$.

8. For each prime number p and each positive integer n, how many elements α are there in $\mathbb{F}_{p^n}$ such that $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^6}$?