Algebra Prelim

September 18, 2007

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The letters k and K always denote fields.

1. Let K be a field of characteristic zero and L a Galois extension of K. Let f be an irreducible polynomial in $K[x]$ of degree 7 and suppose f has no zeroes in L. Show that f is irreducible in $L[x]$.

2. Let K be a field of characteristic zero and $f \in K[x]$ an irreducible polynomial of degree n. Let L be a splitting field for f. Let G be the group of automorphisms of L which act trivially on K.

   (a) Show that G embeds in the symmetric group S_n.

   (b) For each n, give an example of a field K and polynomial f such that $G = S_n$.

   (c) What are the possible groups G when $n = 3$? Justify your answer.

3. Show there are exactly two groups of order 21 up to isomorphism.

4. (a) Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a unique factorisation domain (UFD).

   (b) Is $\mathbb{Z}[\sqrt{-5}]$ a UFD? Justify your answer.

5. Let A be a domain and K its field of fractions. Recall that we say $f \in K$ is *integral over* A if it satisfies an equation

$$f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = 0,$$

where $a_{n-1}, \ldots, a_0 \in A$. The *integral closure* $\tilde{A} \subset K$ of A is the set of $f \in K$ which are integral over A, and we say A is *integrally closed* if $\tilde{A} = A$.

   (a) Show that a UFD is integrally closed. (Hint: write f as a fraction.)

   (b) Compute the integral closure of $k[x, y]/(x^2 - y^3)$. (Remember that a polynomial ring is a UFD and therefore integrally closed.)

   (c) Compute the integral closure of $k[x, y, z]/(x^2 - y^2 z)$. (Hint: there is an obvious integral element.)

6. Let V be a finite dimensional vector space over $\mathbb{Q}$ and $A \colon V \to V$ a linear map such that $A^7 = \mathrm{id}$, the identity map. Suppose that 1 is not an eigenvalue of A. Prove that $\dim V$ is divisible by 6.

7. Let V be a vector space over a field k that is not of characteristic two. Let $\omega \colon V \times V \to k$ be a non-degenerate skew-symmetric bilinear form, i.e., $\omega(x, y) = -\omega(y, x)$ for all $x, y \in V$, and if $x \neq 0$ there is a y such that $\omega(x, y) \neq 0$.

   (a) Show that there exists a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of V such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$ for all i, j. (In particular $\dim V = 2n$ is even.)

   (b) We say a subspace $W \subset V$ is *isotropic* if $\omega(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. Show that the dimension of an isotropic subspace is at most $\frac{1}{2} \dim V$.

8. Let $\mathbb{H}$ be the ring of quaternions with standard basis $1, i, j, k$ and identify $\mathbb{C}$ with the subring $\mathbb{R} + \mathbb{R}i$ of $\mathbb{H}$.

   (a) Use the action of $\mathbb{H}$ on itself by left multiplication to explain why there is a ring homomorphism $\varphi : \mathbb{H} \to M_2(\mathbb{C})$, where $M_2(\mathbb{C})$ denotes the ring of 2×2 matrices. (Warning: there are two ways to view $\mathbb{H}$ as a $\mathbb{C}$-vector space, through right and left multiplication by elements in the subring $\mathbb{R} + \mathbb{R}i$.)

(b) Say why φ is injective.

(c) The special unitary group SU(2) consists of all 2×2 complex matrices u such that $\det(u) = 1$ and $uu^* = u^*u = 1$ where u^* is the conjugate transpose, i.e., the transpose of the matrix whose entries are the complex conjugates of the entries in u. Show that φ restricts to an isomorphism between the group of unit quaternions (those of length one) and SU(2).

(d) Use this to prove that SU(2) acts transitively on the Riemann sphere $\mathbb{CP}^1$ defined as the 1-dimensional subspaces in $\mathbb{C}^2$. (Hint: use the action of $M_2(\mathbb{C})$ on $\mathbb{C}^2$ by left multiplication.)

(e) Let $U(1)$ denote the image in SU(2) of the multiplicative subgroup of $\mathbb{C} - \{0\}$ consisting of the complex numbers $z \in \mathbb{C} \subset \mathbb{H}$ of length one. Show that the coset space $\mathrm{SU}(2)/U(1)$ is isomorphic to the 2-sphere S^2.

Remark. The solution to this problem gives a realization of the Hopf fibration $S^3 \to S^2$ with fibers S^1 because the group of unit quaternions is isomorphic to the 3-sphere S^3.