Algebra Preliminary Exam

September 8, 2008

Instructions: *Do as many problems as you can. Single complete solutions are better than several partial solutions; correct answers to four problems are sufficient to pass. Do not reprove major theorems unless asked to do so, but when you use such theorems say so. In writing down partial solutions try to indicate the gaps as clearly as possible, so that we can see what you do and don't know.*

1. Let $f(x)$ be an irreducible polynomial of degree 5 over the field $\mathbb{Q}$ of rational numbers with exactly 3 real roots.
   (a) Show that $f(x)$ is not solvable by radicals.
   (b) Let E be the splitting field of f over $\mathbb{Q}$. Construct a Galois extension K of degree 2 over $\mathbb{Q}$ lying in E such that *no* field F strictly between K and E is Galois over $\mathbb{Q}$.

2. Let F be a finite field. Show for any positive integer n that there are irreducible polynomials of degree n in $F[x]$.

3. Show that the order of the group $GL_n(\mathbb{F}_q)$ of invertible $n \times n$ matrices over the field $\mathbb{F}_q$ of q elements is given by $(q^n - 1)(q^n - q)\ldots(q^n - q^{n-1})$.

4. By looking at degrees of polynomials, show that any $\mathbb{C}$-subalgebra of the ring $\mathbb{C}[x]$ of polynomials in one variable over the complex field $\mathbb{C}$ is finitely generated.

5.(a) Let R be a commutative principal ideal domain. Show that any R-module M generated by two elements takes the form $R/(a) \oplus R/(b)$ for some $a, b \in R$. What more can you say about a and b?
   (b) Give a necessary and sufficient condition for two direct sums as in part (a) to be isomorphic as R-modules.

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6. Let G be the subgroup of $GL_3(\mathbb{C})$ generated by the three matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

where $i^2 = -1$. Here $\mathbb{C}$ denotes the complex field.

(a) Compute the order of G.

(b) Find a matrix in G of largest possible order (as an element of G) and compute this order.

(c) Compute the number of elements in G with this largest order.

7.(a) Let G be a group of (finite) order n. Show that any irreducible left module over the group algebra $\mathbb{C}G$ has complex dimension at most $\sqrt{n}$.

(b) Give an example of a group G of order $n \geq 5$ and an irreducible left module over $\mathbb{C}G$ of complex dimension $\lfloor \sqrt{n} \rfloor$, the greatest integer to $\sqrt{n}$.

8. Use the rational canonical form to show that any square matrix M over a field k is similar to its transpose M^t, recalling that $p(M) = 0$ for some $p \in k[t]$ if and only if $p(M^t) = 0$.