2010 Algebra Prelim

August 31, 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count for more than several partial solutions. Always justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Let p be a positive prime number, $\mathbb{F}_p$ the field with p elements, and let $G = \mathrm{GL}_2(\mathbb{F}_p)$.
   (a) Compute the order of G, $|G|$.
   (b) Write down an explicit isomorphism from $\mathbb{Z}/p\mathbb{Z}$ to
   $$U = \left\{ \left(\begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \middle| \, a \in \mathbb{F}_p \right\}.$$
   (c) How many subgroups of order p does G have?
   Hint: compute gug^{-1} for $g \in G$ and $u \in U$; use this to find the size of the normalizer of U in G.

2. (a) Give definitions of the following terms: (i) a finite length (left) module, (ii) a composition series for a module, and (iii) the length of a module,
   (b) Let $l(M)$ denote the length of a module M. Prove that if
   $$0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$$
   is an exact sequence of modules of finite length, then
   $$\sum_{i=1}^{n} (-1)^i l(M_i) = 0.$$

3. Let $\mathbb{F}$ be a field of characteristic p, and G a group of order p^n. Let $R = \mathbb{F}[G]$ be the group ring (group algebra) of G over $\mathbb{F}$, and let $u := \sum_{x \in G} x$ (so u is an element of R).
   (a) Prove that u lies in the center of R.

(b) Verify that Ru is a 2-sided ideal of R.

(c) Show there exists a positive integer k such that $u^k = 0$. Conclude that for such a k, $(Ru)^k = 0$.

(d) Show that R is **not** a semi-simple ring. (**Warning:** Please use the definition of a semisimple ring; do **not** use the result that a finite length ring fails to be semisimple if and only if it has a non-zero nilpotent ideal.)

4. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ (where $a_n \neq 0$) and let $R = \mathbb{Z}[x]/(f)$. Prove that R is a finitely-generated module over $\mathbb{Z}$ if and only if $a_n = \pm 1$.

5. Consider the ring

$$S = C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$

with the usual operations of addition and multiplication of functions.

(a) What are the invertible elements of S?

(b) For $a \in [0,1]$, define $I_a = \{f \in S \mid f(a) = 0\}$. Show that I_a is a maximal ideal of S.

(c) Show that the elements of any proper ideal of S have a common zero, i.e., if I is a proper ideal of S, then there exists $a \in [0,1]$ such that $f(a) = 0$ for all $f \in I$. Conclude that every maximal ideal of S is of the form I_a for some $a \in [0,1]$.
Hint: as $[0,1]$ is compact, every open cover of $[0,1]$ contains a finite subcover.

6. (a) Let L/F be a field extension that is finite and Galois. Show that if the Galois group $\mathrm{Gal}(L/F)$ is abelian then for every intermediate field $F \subseteq K \subseteq L$, K/F is also a Galois extension.

(b) Let $K = \mathbb{Q}\left(\sqrt{1+\sqrt{2}}\right) \subset \mathbb{R}$. Show that $K/\mathbb{Q}$ is an extension of degree 4 that is **not** Galois.

(c) Let L be the smallest Galois extension of $\mathbb{Q}$ that contains $K = \mathbb{Q}\left(\sqrt{1+\sqrt{2}}\right)$. Compute the group $\mathrm{Gal}(L/\mathbb{Q})$.

7. Let F be a field of characteristic zero, and let K be an **algebraic** extension of F that possesses the following property: every polynomial $f \in F[x]$ has a root in K. Show that K is algebraically closed.
Hint: if $K(\theta)/K$ is algebraic, consider $F(\theta)/F$ and its normal closure; primitive elements might be of help.

8. Let G be the unique non-abelian group of order 21.

(a) Describe all 1-dimensional complex representations of G.

(b) How many (non-isomorphic) irreducible complex representations does G have and what are their dimensions?

(c) Determine the character table of G.