2010 Algebra Prelim

August 31, 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count for more than several partial solutions. Always justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Let \( p \) be a positive prime number, \( \mathbb{F}_p \) the field with \( p \) elements, and let \( G = \text{GL}_2(\mathbb{F}_p) \).

   (a) Compute the order of \( G \), \(|G|\).

   (b) Write down an explicit isomorphism from \( \mathbb{Z}/p\mathbb{Z} \) to

   \[
   U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \bigg| a \in \mathbb{F}_p \right\}.
   \]

   (c) How many subgroups of order \( p \) does \( G \) have?

   **Hint:** compute \( gug^{-1} \) for \( g \in G \) and \( u \in U \); use this to find the size of the normalizer of \( U \) in \( G \).

2. (a) Give definitions of the following terms: (i) a finite length (left) module, (ii) a composition series for a module, and (iii) the length of a module,

   (b) Let \( l(M) \) denote the length of a module \( M \). Prove that if

   \[
   0 \to M_1 \to M_2 \to \cdots \to M_n \to 0
   \]

   is an exact sequence of modules of finite length, then

   \[
   \sum_{i=1}^{n} (-1)^i l(M_i) = 0.
   \]

3. Let \( \mathbb{F} \) be a field of characteristic \( p \), and \( G \) a group of order \( p^n \). Let \( R = \mathbb{F}[G] \) be the group ring (group algebra) of \( G \) over \( \mathbb{F} \), and let \( u := \sum_{x \in G} x \) (so \( u \) is an element of \( R \)).

   (a) Prove that \( u \) lies in the center of \( R \).
(b) Verify that $Ru$ is a 2-sided ideal of $R$.
(c) Show there exists a positive integer $k$ such that $u^k = 0$. Conclude that for such a $k$, $(Ru)^k = 0$.
(d) Show that $R$ is not a semi-simple ring. (Warning: Please use the definition of a semisimple ring; do not use the result that a finite length ring fails to be semisimple if and only if it has a non-zero nilpotent ideal.)

4. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ (where $a_n \neq 0$) and let $R = \mathbb{Z}[x]/(f)$. Prove that $R$ is a finitely-generated module over $\mathbb{Z}$ if and only if $a_n = \pm 1$.

5. Consider the ring
$$S = C[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$$
with the usual operations of addition and multiplication of functions.

(a) What are the invertible elements of $S$?
(b) For $a \in [0,1]$, define $I_a = \{f \in S \mid f(a) = 0\}$. Show that $I_a$ is a maximal ideal of $S$.
(c) Show that the elements of any proper ideal of $S$ have a common zero, i.e., if $I$ is a proper ideal of $S$, then there exists $a \in [0,1]$ such that $f(a) = 0$ for all $f \in I$. Conclude that every maximal ideal of $S$ is of the form $I_a$ for some $a \in [0,1]$.
   Hint: as $[0,1]$ is compact, every open cover of $[0,1]$ contains a finite subcover.

6. (a) Let $L/F$ be a field extension that is finite and Galois. Show that if the Galois group $\text{Gal}(L/F)$ is abelian then for every intermediate field $F \subseteq K \subseteq L$, $K/F$ is also a Galois extension.
(b) Let $K = \mathbb{Q}\left(\sqrt{1 + \sqrt{2}}\right) \subset \mathbb{R}$. Show that $K/Q$ is an extension of degree 4 that is not Galois.
(c) Let $L$ be the smallest Galois extension of $\mathbb{Q}$ that contains $K = \mathbb{Q}\left(\sqrt{1 + \sqrt{2}}\right)$. Compute the group $\text{Gal}(L/Q)$.

7. Let $F$ be a field of characteristic zero, and let $K$ be an algebraic extension of $F$ that possesses the following property: every polynomial $f \in F[x]$ has a root in $K$. Show that $K$ is algebraically closed.
   Hint: if $K(\theta)/K$ is algebraic, consider $F(\theta)/F$ and its normal closure; primitive elements might be of help.

8. Let $G$ be the unique non-abelian group of order 21.
   (a) Describe all 1-dimensional complex representations of $G$.
   (b) How many (non-isomorphic) irreducible complex representations does $G$ have and what are their dimensions?
   (c) Determine the character table of $G$. 