Algebra Prelim
September 12, 2011

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Let $\mathrm{GL}_2(\mathbb{C})$ be the general linear group of 2×2 complex matrices, let H be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ consisting of non-zero multiples of the identity matrix, and let $\mathrm{PGL}_2(\mathbb{C})$ be the quotient group $\mathrm{GL}_2(\mathbb{C})/\mathrm{H}$.

   Let $A, B \in \mathrm{PGL}_2(\mathbb{C})$, and assume that both elements have order n. Prove that there exist $C \in \mathrm{PGL}_2(\mathbb{C})$ and a positive integer m such that

   $$CBC^{-1} = A^m.$$

2. In this problem, as you apply Sylow's Theorem, state precisely which portions you are using.

   (a) Prove that there is no simple group of order 30.

   (b) Suppose that G is a simple group of order 60. Determine the number of p-Sylow subgroups of G for each prime p dividing 60, then prove that G is isomorphic to the alternating group A_5.

   Note: In the second part, you needn't show that A_5 is simple. You need only show that if there is a simple group of order 60, then it must be isomorphic to A_5.

3. Describe the Galois group and the intermediate fields of the cyclotomic extension $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

4. Let
   $$R = \mathbb{Z}[x]/(x^2 + x + 1).$$

   (a) Answer the following questions with suitable justification.

      i. Is R a Noetherian ring?

      ii. Is R an Artinian ring?

   (b) Prove that R is an integrally closed domain.

5. Let R be a commutative ring. Recall that an element r of R is *nilpotent* if $r^n = 0$ for some positive integer n and that the *nilradical* of R is the set $N(R)$ of nilpotent elements.

   (a) Prove that
   $$N(R) = \bigcap_{P \text{ prime}} P.$$
   (Hint: Given a non-nilpotent element r of R, you may wish to construct a prime ideal that does not contain r or its powers.)

   (b) Given a positive integer m, determine the nilradical of $\mathbb{Z}/(m)$.

   (c) Determine the nilradical of $\mathbb{C}[x, y]/(y^2 - x^3)$.

   (d) Let $p(x, y)$ be a polynomial in $\mathbb{C}[x, y]$ such that for any complex number a, $p(a, a^{3/2}) = 0$. Prove that $p(x, y)$ is divisible by $y^2 - x^3$.

6. Given a finite group G, recall that its *regular representation* is the representation on the complex group algebra $\mathbb{C}[G]$ induced by left multiplication of G on itself and its *adjoint representation* is the representation on the complex group algebra $\mathbb{C}[G]$ induced by conjugation of G on itself.

   (a) Let $G = \mathrm{GL}_2(\mathbb{F}_2)$. Describe the number and dimensions of the irreducible representations of G. Then describe the decomposition of its regular representation as a direct sum of irreducible representations.

   (b) Let H be a group of order 12. Show that its adjoint representation is reducible; that is, there is an H-invariant subspace of $\mathbb{C}[H]$ besides 0 and $\mathbb{C}[H]$.

7. Let M, N be finitely generated modules over $\mathbb{Z}$. Recall that $\mathrm{Ann}(M)$ is the ideal in $\mathbb{Z}$ defined as follows:
   $$\mathrm{Ann}(M) = \{a \in \mathbb{Z} \,|\, am = 0 \text{ for any } m \in M\}$$
   Prove that $M \otimes_{\mathbb{Z}} N = 0$ if and only if $\mathrm{Ann}(M) + \mathrm{Ann}(N) = (1)$.

8. Let R be a commutative integral domain. Show that the following are equivalent:

   (a) R is a field;

   (b) R is a semi-simple ring;

   (c) Any R-module is projective.