Algebra Prelim

September 10, 2012

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Classify all groups of order 2012 up to isomorphism. (Hint: 503 is prime.)

2. For any positive integer n, let G_n be the group generated by a and b subject to the following three relations:
$$a^2 = 1, \quad b^2 = 1, \quad \text{and} \quad (ab)^n = 1.$$
   (a) Find the order of the group G_n.
   (b) Classify all irreducible complex representations of G_4 up to isomorphism.

3. Let R be a (commutative) principal ideal domain, let M and N be finitely generated free R-modules, and let $\varphi : M \to N$ be an R-module homomorphism.
   (a) Let K be the kernel of φ. Prove that K is a direct summand of M.
   (b) Let C be the image of φ. Show by example (specifying R, M, N and φ) that C need not be a direct summand of N.

4. Let G be an abelian group. Prove that the group ring $\mathbb{Z}[G]$ is noetherian if and only if G is finitely generated.

5. Let $M_3(\mathbb{R})$ be the 3×3-matrix algebra over the real numbers $\mathbb{R}$. For any $b \in \mathbb{R}$, let $B \in M_3(\mathbb{R})$ be the matrix $\begin{pmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{pmatrix}$. Find the set of numbers b so that the matrix equation $X^2 = B$ has at least one, and only finitely many, solutions in $M_3(\mathbb{R})$.

6. Determine the Galois groups of the following polynomials over $\mathbb{Q}$.
   (a) $f(x) = x^4 + 4x^2 + 1$
   (b) $f(x) = x^4 + 4x^2 - 5$

7. Prove that if A is a finite abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong A$. (Here, $\mathrm{Ext}^1_{\mathbb{Z}}(-,-)$ is also sometimes written as $\mathrm{Ext}(-,-)$.)

8. Let A be the $\mathbb{C}$-algebra $\mathbb{C}[x, y]$, and define algebra automorphisms σ and τ of A by
$$\sigma(x) = y, \quad \sigma(y) = x$$
and
$$\tau(x) = x, \quad \tau(y) = \xi y,$$
where $\xi \in \mathbb{C}$ is a primitive third root of unity (namely, $\xi \neq 1$ and $\xi^3 = 1$). Let G be the group of algebra automorphisms of A generated by σ and τ. Define
$$A^G = \{f \in A \mid \phi(f) = f \text{ for all } \phi \in G\}.$$
Then A^G is a subalgebra of A – you do not need to prove this. Describe the algebra A^G by finding a set of generators and a set of relations.