INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation: Q is the field of rational numbers and C is the field of complex numbers.

Problem 1. Let $\mathbb{Q}^\times$ be the nonzero elements of $\mathbb{Q}$, a group under multiplication.
(a) Prove that the additive group of $\mathbb{Q}$ has no maximal proper subgroups.
(b) Is the same statement true for the multiplicative group $\mathbb{Q}^\times$?

Problem 2. Let $V$ be a finite-dimensional vector space over a field $F$ of characteristic 0. Let $B : V \times V \to F$ be a non-degenerate, skew-symmetric bilinear form. (In particular, we have $B(x, y) = -B(y, x)$ for all $x, y \in V$.) If $U$ is a subset of $V$, let
$$U^\perp = \{ v \in V \mid B(u, v) = 0 \text{ for all } u \in U \}.$$ 
(a) Let $U$ be a subspace of $V$. Prove that $U^\perp$ is a subspace of $V$ and that
$$\dim_F(U) + \dim_F(U^\perp) = \dim_F(V).$$
(b) Prove that there exists a subspace $W$ of $V$ such that $W^\perp = W$.

Problem 3.
(a) Suppose that $G$ is a finitely-generated group. Let $n$ be a positive integer. Prove that $G$ has only finitely many subgroups of index $n$.
(b) Let $p$ be a prime number. If $G$ is any finitely-generated abelian group, let $t_p(G)$ denote the number of subgroups of $G$ of index $p$. Determine the possible values of $t_p(G)$ as $G$ varies over all finitely-generated abelian groups.

Problem 4. Suppose that $G$ is a finite group of order 2013. Prove that $G$ has a normal subgroup $N$ of index 3 and that $N$ is a cyclic group. Furthermore, prove that the center of $G$ has order divisible by 11. (You will need the factorization $2013 = 3 \cdot 11 \cdot 61.$)
Problem 5. Let $V$ be a finite dimensional vector space over $\mathbb{C}$. Let $n = \text{dim}_\mathbb{C}(V)$. Let $T : V \to V$ be a linear map. Suppose that the following statement is true.

For every $c \in \mathbb{C}$, the subspace $\{ v \in V \mid T(v) = cv \}$ of $V$ has dimension 0 or 1.

Prove that there exists a vector $w \in V$ such that $\{w, T(w), \ldots, T^{n-1}(w)\}$ is a linearly independent set.

Problem 6. This question concerns an extension $K$ of $\mathbb{Q}$ such that $[K : \mathbb{Q}] = 8$. Assume that $K/\mathbb{Q}$ is Galois and let $G = \text{Gal}(K/\mathbb{Q})$. Furthermore, assume that $G$ is nonabelian.

(a) Prove that $K$ has a unique subfield $F$ such that $F/\mathbb{Q}$ is Galois and $[F : \mathbb{Q}] = 4$.

(b) Prove that $F$ has the form $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where $d_1$ and $d_2$ are nonzero integers.

(c) Suppose that $G$ is the quaternionic group. Prove that $d_1$ and $d_2$ are positive integers.

Problem 7. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{C}$ in $n$ indeterminates $x_1, \ldots, x_n$. Let $S_n$ be the $n$-th symmetric group. If $\sigma \in S_n$, then we can identify $\sigma$ with the automorphism of $R$ defined as follows: $\sigma(c) = c$ for all $c \in \mathbb{C}$, and $\sigma(x_i) = x_{\sigma(i)}$ for all $i$, $1 \leq i \leq n$. Suppose that $G$ is any subgroup of $S_n$. Let

$$S = R^G = \{ r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G \} .$$

Prove that $S$ is a finitely-generated $\mathbb{C}$-algebra.

Problem 8. This question concerns the polynomial ring $R = \mathbb{Z}[x, y]$ and the ideal $I = (5, x^2 + 2)$ in $R$.

(a) Prove that $I$ is a prime ideal of $R$ and that $R/I$ is a PID.

(b) Give an explicit example of a maximal ideal of $R$ which contains $I$.

(Give a set of generators for such an ideal.)

(c) Show that there are infinitely many distinct maximal ideals in $R$ which contain $I$. 