2014 Algebra Prelim

September 8, 2014

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count for more than several partial solutions. Always justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. (a) Let G be a group (not necessarily finite) that contains a subgroup of index n. Show that G contains a *normal* subgroup N such that $n \leq [G : N] \leq n!$.

   (b) Use part (a) to show that there is no simple group of order 36.

2. Let p be a prime, let $\mathbb{F}_p$ be the p-element field, and let $K = \mathbb{F}_p(t)$ be the field of rational functions in t with coefficients in $\mathbb{F}_p$. Consider the polynomial $f(X) = X^p - t \in K[X]$.

   (a) Show that f does not have a root in K.

   (b) Let E be the splitting field of f over K. Find the factorization of f over E.

   (c) Conclude that f is irreducible over K.

3. Recall that a ring A is called *graded* if it admits a direct sum decomposition $A = \oplus_{n=0}^{\infty} A_n$ as abelian groups, with the property that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$.

   Prove that a graded commutative ring $A = \oplus_{n=0}^{\infty} A_n$ is Noetherian if and only if A_0 is Noetherian and A is finitely generated as an algebra over A_0.

4. Let R be a ring with the property that $a^2 = a$ for all $a \in R$.

   (a) Compute the Jacobson radical of R.

   (b) What is the characteristic of R?

   (c) Prove that R is commutative.

   (d) Prove that if R is finite, then R is isomorphic (as a ring) to $(\mathbb{Z}/2\mathbb{Z})^d$ for some d.

5. Let R be a commutative ring and let M be an R-module.

   (a) Let $x \in R$ be a nonzero divisor. Compute $\mathrm{Tor}_i^R(R/(x), M)$ for $i \geq 0$.

(b) Show that M is a *flat* R-module if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N.

(c) Conclude that if R is a PID and M is a finitely generated R-module, then M is flat if and only if M is free.

6. Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. Show that the Galois group $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ has no nontrivial finite subgroups.

7. Let C_p denote the cyclic group of prime order p.

(a) Show that C_p has two irreducible representations over $\mathbb{Q}$ (up to isomorphism), one of dimension 1 and one of dimension $p - 1$.

(b) Let G be a finite group, and let $\rho : G \longrightarrow \mathrm{GL}_n(\mathbb{Q})$ be a representation of G over $\mathbb{Q}$. Let $\rho_{\mathbb{C}} : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ denote ρ followed by the inclusion $\mathrm{GL}_n(\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$. Thus $\rho_{\mathbb{C}}$ is a representation of G over $\mathbb{C}$, called the complexification of ρ. We say that an irreducible representation ρ of G is *absolutely irreducible* if its complexification remains irreducible over $\mathbb{C}$.

Now suppose G is abelian and that every representation of G over $\mathbb{Q}$ is absolutely irreducible. Show that $G \cong (C_2)^k$ for some k (i.e., is a product of cyclic groups of order 2).

8. Let G be a finite group and $\mathbb{Z}[G]$ the integral group algebra. Let $\mathcal{Z}$ be the center of $\mathbb{Z}[G]$. For each conjugacy class $C \subset G$, let $P_C = \sum_{g \in C} g$.

(a) Show that the elements P_C form a $\mathbb{Z}$-basis for $\mathcal{Z}$. Hence $\mathcal{Z} \cong \mathbb{Z}^d$ as an abelian group, where d is the number of conjugacy classes in G.

(b) Show that if a ring R is isomorphic to $\mathbb{Z}^d$ as an abelian group, then every element in R satisfies a monic integral polynomial. (**Hint:** Let $\{v_1, \ldots, v_d\}$ be a basis of R and for a fixed non-zero $r \in R$, write $rv_i = \sum_j a_{ij} v_j$. Use the Hamilton-Cayley theorem.)

(c) Let $\pi : G \longrightarrow \mathrm{GL}(V)$ be an irreducible representation of G (over $\mathbb{C}$). Show that $\pi(P_C)$ acts on V as multiplication by the scalar

$$\frac{|C| \chi_\pi(C)}{\dim V},$$

where $\chi_\pi(C)$ is the value of the character χ_π on any element of C.

(d) Conclude that $|C|\chi_\pi(C)/\dim V$ is an algebraic integer.