ALGEBRA PRELIMINARY EXAM – AUTUMN 2016

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count for more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Let G be a finite simple group (that is, a group with no proper nontrivial normal subgroup). Assume that every proper subgroup of G is abelian. Prove that then G is cyclic of prime order.

2. Let $a \in \mathbb{N}$, $a > 0$. Compute the Galois group of the splitting field of the polynomial $x^5 - 5a^4x + a$ over $\mathbb{Q}$.

3. Let $\mathfrak{m} \subset \mathbb{Z}[x_1, \ldots, x_n]$ be a maximal ideal. Show that $\mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field.

4. Recall that an inner automorphism of a group is an automorphism given by conjugation by an element of the group. An outer automorphism is an automorphism that is not inner.

(a) Prove that S_5 has a subgroup of order 20.

(b) Use the subgroup from (a) to construct a degree 6 permutation representation of S_5 (i.e., an embedding $S_5 \hookrightarrow S_6$ as a transitive permutation group on 6 letters).

(c) Conclude that S_6 has an outer automorphism.

5. Let A be a commutative ring and M a finitely generated A-module. Define

$$\operatorname{Ann}(M) = \{a \in A \mid am = 0 \text{ for all } m \in M\}.$$

Show that for a prime ideal $\mathfrak{p} \subset A$, the following are equivalent:

(a) $\operatorname{Ann} M \not\subset \mathfrak{p}$.

(b) The localization of M at the prime ideal $\mathfrak{p}$ is 0.

(c) $M \otimes_A k(\mathfrak{p}) = 0$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is the residue field of A at $\mathfrak{p}$.

6. Let $A = \mathbb{C}[x, y]/(y^2 - (x-1)^3 - (x-1)^2)$.

(a) Show that A is an integral domain and sketch the $\mathbb{R}$-points of $\operatorname{Spec} A$.

(b) Find the integral closure of A. Recall that for an integral domain A with a fraction field K, the integral closure of A in K is the set of all elements of K integral over A.

7. Let $R = k[x, y]$ where k is a field, and let $I = (x, y)R$.

(a) Show that

$$0 \longrightarrow R \xrightarrow{\phi} R \oplus R \xrightarrow{\psi} R \longrightarrow k \longrightarrow 0$$

where $\phi(a) = (-ya, xa)$, $\psi((a, b)) = xa + yb$ for $a, b \in R$, is a projective resolution of the R-module $k \simeq R/I$.

(b) Show that I is not a flat R-module by computing $\mathrm{Tor}_i^R(I, k)$.

8. Let k be a field of positive characteristic p, $\mathbb{Z}/p$ a cyclic group of order p, and $k\mathbb{Z}/p$ the group algebra of $\mathbb{Z}/p$ over k.

(a) Let σ be a generator of $\mathbb{Z}/p$ and let $t = \sigma - 1$. Show that there is an isomorphism $k\mathbb{Z}/p \simeq k[t]/t^p$.

(b) Let M be a finite dimensional *projective* $k\mathbb{Z}/p$-module and Σ the linear operator on M induced by the action of σ. Show that

(i) $\dim_k M$ is divisible by p; and

(ii) $\mathrm{rk}(\Sigma - \mathrm{Id}_M) = \dfrac{p-1}{p} \dim_k M$.

(c) Let M be a finite dimensional $k\mathbb{Z}/p$-module, and assume that

$$\mathrm{rk}(\Sigma - \mathrm{Id}_M) = \frac{p-1}{p} \dim_k M.$$

Prove that then M is projective.