ALGEBRA PRELIMINARY EXAM – FALL 2020

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count for more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Determine the number of 5-Sylow subgroups of $\mathrm{SL}_2(\mathbf{F}_5)$.

2. Let ζ be a primitive 37th root of unity, and let $\eta = \zeta + \zeta^{10} + \zeta^{26}$. Determine the Galois group the field extension $\mathbb{Q}(\eta)/\mathbb{Q}$.

3. Let $\mathcal{M}_n(\mathbb{C})$ be the set of $n \times n$ matrices with entries in $\mathbb{C}$. For a matrix $A = (a_{ij})$, the (usual) *trace* function $\mathrm{Tr} : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ is given by the formula $\mathrm{Tr}(A) = \sum_i a_{ii}$. Recall that Tr is commutative: for any two $n \times n$ matrices A, B, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. In this problem you will prove that this is a unique such functional, up to a scalar multiplication.

Let $f \colon \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ be a linear functional which has the property $f(AB) = f(BA)$ for all $A, B \in \mathcal{M}_n(\mathbb{C})$. Prove that there exists a constant $c \in \mathbb{C}$ such that $f = c\,\mathrm{Tr}$.

Hint. Show that the linear subspace of $\mathcal{M}_n(\mathbb{C})$ generated by commutators $[A, B] = AB - BA$ has codimension 1.

4. Let G be a finite group. Show that the number of irreducible representations of G is strictly greater than the number of irreducible representations of any of its factor groups by a non-trivial normal subgroup.

5. Find all commutative rings R with 1 such that R has a unique maximal ideal and such that the only units of R are 1 and -1.

6. Let $\Lambda = \mathbb{C}[x]/(x^2)$, and let M be a complex vector space of dimension n which has a structure of a Λ-module. Denote by $\mathrm{End}_{\mathbb{C}}(M)$ the $\mathbb{C}$-algebra of $\mathbb{C}$-linear endomorphisms of M, and by $\rho_M : \Lambda \to \mathrm{End}_{\mathbb{C}}(M)$ the linear map that realizes the structure of Λ-module on M. Recall that by choosing a basis of M one can identify $\mathrm{End}_{\mathbb{C}}(M)$ with $n \times n$ matrices; hence, one can talk about ranks of elements of $\mathrm{End}_{\mathbb{C}}(M)$.

(1) Show that

$$\mathrm{rank}_{\mathbb{C}}(\rho_M(x)) \leq \frac{\dim_{\mathbb{C}} M}{2}.$$

(2) Show that the equality $\mathrm{rank}_{\mathbb{C}}(\rho_M(x)) = \frac{\dim_{\mathbb{C}} M}{2}$ holds if and only if M is a free Λ-module.

7. Let A be a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$. Show that a finitely generated A-module M is free if and only if $\mathrm{Tor}_1^A(A/\mathfrak{m}, M) = 0$.

8. Let k be a field.

(1) Let R be a (commutative) k-algebra, and M be an R-module. Define what it means for M to be
  (a) a projective R-module;
  (b) an injective R-module.
State the definitions you intend to use for the remaining parts of the problem.

(2) Let $R = k[x]/(x^\ell)$ where x is an independent variable. Show that any finitely generated projective R-module is injective.

(3) Let $R = k[x]$ where x is an independent variable. Show that there is no projective R-module (either finitely or infinitely generated) which is also injective.