Complex Analysis Preliminary Exam

Autumn 2005

There are eight problems. Do as many problems as you can. Four completely correct problems will be a clear pass. Complete problems count more than many problem fragments. In all problems, **C** denotes the complex numbers and **D** the unit disc.

1. Compute using residues, where n is a positive even integer:

$$\int_0^\infty \frac{dx}{1+x^n}.$$

   Hint: Find a contour that surrounds only one pole.

2. Let a and b be complex numbers such that $0 < |a| < |b|$. Write down all Taylor and Laurent series of

$$f(z) = \frac{1}{(z-a)(z-b)}$$

   centered at 0, and state where they converge.

3. Let G be a domain and $f_n : G \to \mathbf{C}$ be a sequence of analytic functions such that $f_n(z)$ converges for every $z \in G$. Suppose there are analytic functions g_n with $|g_n| \le 1$ and $|f_n - g_n| \ge 1$ in G. Prove that f_n converges uniformly on compact subsets of G.

4. Let $U = \{z : |z-1| < 2 \text{ and } |z+1| < 2\}$. Find a conformal map $f : U \to \mathbf{D}$. Note: It is acceptable to leave your answer as a composition of conformal maps.

5. Let f be analytic in $H = \{x > 0\}$ and suppose the real part satisfies $0 \le \Re f(x+iy) \le Mx$ for some constant $M > 0$ and all $x + iy \in H$. Show that $f(z) = mz + ic$ for some real constants c and $0 \le m \le M$.

6. Let f be analytic in a disc $D = \{|z| < r\}$ and let $\gamma : [a,b] \to \mathbf{C}$ be a curve (continuous function) with initial point $\gamma(a)$ in D.
   (a) Write down the definition that f has an analytic continuation along γ.
   (b) Now suppose that f has an analytic continuation along every line segment beginning at 0. Prove that f can be extended to a function analytic on **C**.

7. Write down an infinite product that converges to an entire function $f(z)$ with zeroes of order 1 at the points $z_n = \sqrt{n}$, $n = 1, 2, 3, \ldots$, and no other zeroes. Prove the convergence of your product.

8. Let A denote the annulus $\{r_1 < |z| < r_2\}$.
   (a) Construct a harmonic function h on A such that h is continuous up to the boundary and $h = 0$ on $\{|z| = r_1\}$ and $h = 1$ on $\{|z| = r_2\}$.
   (b) Let u be harmonic in A and continuous up to the boundary, and set $m_j = \max_{\{|z|=r_1\}} u(z)$, for $j = 1, 2$. Find the best possible upper bound for $u(z)$ in terms of m_1, m_2, r_1, r_2 and z.