COMPLEX ANALYSIS PRELIMINARY EXAMINATION
Autumn 2007

Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. In the following, $\mathcal{D}$ always denotes the open unit disk.

Problem 1. Calculate

$$\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}\,\mathrm{d}x \qquad \text{for } a > 0.$$

Problem 2. Find all conformal maps from the open unit disk $\mathcal{D}$ onto the region (see Figure 1)

$$U := \{z \;:\; |z| < 1 \text{ and } \left|z - \frac{1}{2}\right| > \frac{1}{2}\}\ .$$

You may write the answer as a composition of explicitly written maps.

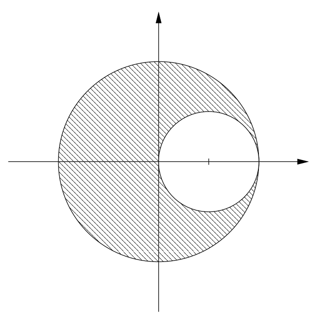

Figure 1: The shaded area represents region U

Problem 3. Let $p(z) = az^4 + bz + 1$ for $a, b \in \mathbb{R}$. Find the maximum number of roots of p in the annulus $\mathcal{A} := \{z \;:\; 1 < |z| < 2\}$, provided that $a \in [1, \pi]$ and $b \in [2\pi - 2, 7]$.

Problem 4. Is there a one-to-one analytic map from the annulus $\Omega_1 := \{z \;:\; \frac{1}{2} < |z| < 1\}$ onto the punctured disk $\Omega_2 := \{0 < |z| < 1\}$? Justify your answer (this is not a yes/no question).

Problem 5.

a) State and prove Schwarz's Lemma, including the case of equality.

b) Given $f : \mathcal{D} \to \mathcal{D}$ an analytic function, show that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \ , \quad \forall \, z \in \mathcal{D} \ .$$

Problem 6. Let M_n, for all $n \in \{0, 1, 2, \ldots\}$ be a sequence of positive numbers, and define the family of functions

$$\mathcal{F} := \{ f \ : \ f \text{ is analytic on } \mathcal{D} \text{ and } \ |f^n(0)| \leq M_n \ , \ \ \forall \, n \in \{0, 1, 2, \ldots\} \} \ .$$

Show that $\mathcal{F}$ is a normal family if and only if $\sum_{n=0}^{\infty} \frac{M_n z^n}{n!}$ converges on $\mathcal{D}$.

Problem 7. Prove that the function $f(z) = \sin(z) - z^2$ has infinitely many complex zeros.

Problem 8. Given a subharmonic function $u : \mathbb{C} \to \mathbb{R}$, recall that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \ \mathrm{d}\theta \ ,$$

for any z and for any $r > 0$.

Let u be a subharmonic function, and let $M(r) = \max_{|z|=r} u(z)$.

a) Prove that for any $0 < r_1 \leq |z| \leq r_2$,

$$u(z) \leq \frac{\log r_2 - \log |z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log |z| - \log r_1}{\log r_2 - \log r_1} M(r_2) \ .$$

b) Show that $\lim_{r \to \infty} \frac{M(r)}{\log r}$ exists (possibly infinite).