Complex Analysis Preliminary Exam

Autumn 2008

Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Many theorems in complex analysis have several different forms. Make sure that it is clear what the hypotheses of the theorem you are using are and check that you have satisfied all of them.

Notation:

- A **region** is a connected open subset of the complex plane $\mathbb{C}$.
- For $r > 0$, $\mathbb{D}_r$ denotes the (open) disc of radius r and center 0: $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$.
- $\mathbb{N}$ denotes the positive integers.

1. Evaluate
$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx.$$
Justify completely any manipulations you make.

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire, non-constant function satisfying
$$|f(z)| \leq e^{\sqrt{|z|}} \quad \text{for all} \quad z \in \mathbb{C}.$$
For $R > 0$, let $n(R)$ equal the number of zeroes of f having modulus less than or equal to R. Prove that there exist non-negative constants A and B such that $\forall R > 0$,
$$n(R) \leq A + B\sqrt{R}.$$
An example of such a function is $\cos(\sqrt{z})$.

3. Let $\mathcal{H}$ be the class of all analytic functions f on $\mathbb{D}_1$ satisfying $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq 100$ for all $z \in \mathbb{D}_1$. Prove that there is a constant $r > 0$ so that for any $f \in \mathcal{H}$, the image of the unit disk under f contains the disk $\mathbb{D}_r$.

4. Suppose f is analytic on a bounded region U in $\mathbb{C}$, and continuous on the closure of U. Suppose that $|f(z)| = 1$ whenever $z \in \partial U$. Prove that either f is constant, or $f(U) = \mathbb{D}_1$.

5. Let $\langle f_n \rangle$ be a sequence of analytic one-to-one functions on a region G. Suppose that $f_n \to f$ uniformly on compact subsets of G. Show that f is analytic on G, and it is either one-to-one or constant on G.

6. Suppose U is a bounded region in $\mathbb{C}$, and $w_0 \in U$. Set

$$\mathcal{F} = \{f : \mathbb{D}_1 \to U : f \text{ is analytic, one-to-one and } f(0) = w_0\}.$$

Set

$$M = \sup_{f \in \mathcal{F}} |f'(0)|.$$

Prove that there exists an $f \in \mathcal{F}$ for which $|f'(0)| = M$ (in particular $M < \infty$), and that for any such f, if

$$f(\mathbb{D}_1) \subset V \subset U,$$

and V is a simply connected region, then $f(\mathbb{D}_1) = V$.

7. Show that the function

$$w = \log\left(\frac{z+1}{z-1}\right) + \frac{2z}{z^2+1}$$

maps $\mathbb{D}_1$ one-to-one and onto the full w-plane with four half-lines deleted. Find the locations of the four end points of the four half-lines. You will need to choose a branch of logarithm.

8. Suppose U is a region in $\mathbb{C}$, and $z_0 \in U$. Suppose $\langle f_n \rangle$ is a sequence of analytic functions on U satisfying:

   (a) $Re(f_n(z)) \leq Re(f_{n+1}(z))$ for all $z \in U$ and $n \in \mathbb{N}$; and

   (b) $\lim_{n\to\infty} f_n(z_0)$ exists in $\mathbb{C}$.

   Prove that $\langle f_n \rangle$ converges uniformly on compact subsets of U.