Complex Analysis Preliminary Exam

Autumn 2009

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

In all problems, $\mathbb{C}$ denotes the complex numbers, $\mathbb{D}$ the unit disc, and $\mathbb{H}$ the right half-plane. A domain is an open connected set.

1. Let Re $z > 0$. Use an appropriate contour to show that

$$\int_0^\infty \frac{e^{-t} - e^{-tz}}{t} dt = \log z.$$

2. If $a > 1$, prove that $z + e^{-z}$ assumes the value a exactly once in $\mathbb{H}$.

3. Let f be an entire function with the property that for every z there is n such that the n-th derivative $f^{(n)}(z) = 0$. Show that f is a polynomial.

4. If D and D' are open discs with $\overline{D'} \subset D$, show that there is $R > 1$ and a linear fractional transformation f that maps $D \setminus \overline{D'}$ conformally onto $\{1 < |z| < R\}$.

5. Let $I = [-1, 1]$ and $W = \{z \in \mathbb{C} : |z| < 2\}$.

(a) Show that every bounded harmonic function in $W \setminus \{0\}$ extends continuously to W. (If you already know a theorem to this effect, state and prove it).

(b) Explicitly construct a bounded harmonic function u on $W \setminus I$ that does not extend continuously to W.

6. (a) If $f : \mathbb{D} \to \mathbb{H}$ is analytic, show that $|f'(0)| \leq 2$ Re $f(0)$.

(b) If $f : \mathbb{D} \to \mathbb{D} \setminus \{0\}$ is analytic, show that $|f'(0)| \leq \frac{2}{e}$.

7. (a) Prove that $\prod_{k=1}^{\infty}(1 + \frac{i}{k})$ diverges but that $\prod_{k=1}^{\infty} |1 + \frac{i}{k}|$ converges.

(b) Construct an infinite product that is an entire function with simple zeroes precisely at the positive integers, and prove convergence from first principles.

8. Let D be a bounded domain and $f : D \to D$ analytic.

If f has a fixed point $f(z_0) = z_0$ with $|f'(z_0)| < 1$, show that the sequence of iterates $f_n = f \circ f \circ \cdots \circ f$ (n compositions) converges uniformly on compact subsets of D.