Complex Prelim, September 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Compute

$$\int_0^\infty \frac{dx}{x^c(x+1)}$$

for each $c \in (0,1)$. Justify all your steps.

2. (a) Construct a conformal mapping of the upper half-plane onto the angle region with an interval removed:

$$\Omega = \{z \in \mathbb{C} : \ z \neq 0, \ \arg z \in (-\pi/4, \pi/4)\} \setminus \{z : \ \Im(z) = 0, \ 0 \leq \Re(z) \leq 1\}.$$

(b) f is a bounded harmonic function on Ω such that f is continuous on $\partial\Omega \setminus \{0\}$ and

$$f(z) = \begin{cases} 0, & \text{if } \arg z = \pi/4; \\ 1, & \text{if } \arg z = -\pi/4; \\ 3, & \text{on } \{z : \ \Im(z) = 0, \ 0 \leq \Re(z) \leq 1\}. \end{cases}$$

Find an expression for $f(3)$.

3. Let f be analytic in the closed unit disc, with $f(-\log 2) = 0$ and

$$|f(z)| \leq |e^z| \quad \text{for all } z \text{ with } |z| = 1.$$

How large can $|f(\log 2)|$ be? Find the best possible upper bound.

4. How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half-plane?

5. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Find an explicit series representation for a meromorphic function on $\mathbb{C}$, which is holomorphic on $\mathbb{C} - \{\mathbb{N}\}$, and which has at each point $n \in \mathbb{N}$ a simple pole with residue n. Include proofs of all required convergence statements.

6. Let $g(z) = u + iv$ be holomorphic in the domain $\Omega = \{z \in \mathbb{C} : |z| < 1, y > 0\}$ and continuous in the closure $\overline{\Omega}$. Assume that

(i) $u = 0$ for $y = 0$ and $x > 0$;

(ii) $v = 0$ for $y = 0$ and $x < 0$.

Show that

$$\frac{|g(z)|}{|z|^{1/2}} \text{ is bounded in } \Omega.$$

7. Let $\mathcal{F}$ be the set of functions f holomorphic in the unit disc $\mathbb{D}$ such that

$$\iint_{|z|<1} |f(x+iy)|^2 \, dxdy \leq 1.$$

Prove that $\mathcal{F}$ is a normal family in $\mathbb{D}$.

8. Define

$$F(z) = \int_0^\infty x^{z-1} e^{-x^2} dx$$

for $\Re(z) > 0$.

(a) Prove that F is an analytic function in the right half-plane $\{z : \Re(z) > 0\}$.

(b) Prove that F extends to a meromorphic function on the whole complex plane.

(c) Find all poles of F and find the singular parts of F at these poles.