Complex Analysis Preliminary Exam

Autumn 2011

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. In all of these problems $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk and $\mathbb{R} = \{z : \text{Im}z = 0\}$ is the real line.

1. (a) Carefully state the Riemann mapping theorem.
   (b) Part of a standard proof is to show that conformal maps *into* $\mathbb{D}$ exist. Prove this fact.

2. For which complex numbers α does $\prod_{n=1}^{\infty} \cos \frac{1}{n^\alpha}$ converge absolutely? (Prove your answer directly from the definition of absolute convergence).

3. Suppose f is entire and $|f(z)| = 1$ for all $z \in \mathbb{R}$. Prove that $f(z) = e^{g(z)}$ where g is entire.

4. Suppose f is analytic on $\mathbb{D}$ and one-to-one on $\{z : 1/2 < |z| < 1\}$. Prove that f is one-to-one on $\mathbb{D}$.

5. Prove that if u is bounded and harmonic on a punctured disk $B(z_0, R) \setminus \{z_0\}$ then u has a harmonic extension to the full open disk $B(z_0, R)$.

6. Give an explicit example of a non-constant *bounded* analytic function on $\mathbb{D}$ such that each point of $\partial\mathbb{D}$ is a limit point of zeros of f. If you use an infinite sum or product, prove it converges.

7. (a) Find a one-to-one analytic map of $\mathbb{D}$ onto $\{(x, y) : y < x^2\}$. Hint: what is the image of the line $\{z : \text{Re}z = 1\}$ by the map z^2?
   (b) Find a one-to-one analytic map of $\mathbb{D}$ onto $\{(x, y) : y > x^2\}$. Hint: First find a region that maps to half of the inside of the parabola.
   You may write your answers to (a) and (b) as a composition $g_1 \circ g_2 \circ \cdots \circ g_n$ of finitely many explicit conformal maps, or their inverses. You may state $g_j = f_j^{-1}$, if the map f_j is explicit.

8. A holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk is called a Bloch function if

$$||f||_{\mathcal{B}} = \sup_{|z|<1} (1 - |z|^2)|f'(z)| < \infty.$$

   Find a constant C, independent of f, so that

$$\sup_{n \geq 1} |a_n| \leq C||f||_{\mathcal{B}}.$$