Complex Analysis Preliminary Exam

Autumn 2012

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

In all problems, $\mathbb{C}$ denotes the complex numbers and $\mathbb{D}$ the open unit disc. A domain is an open connected set.

1. Is there a conformal map from the slit strip

$$D = \{x + iy : -1 < y < 1\} \setminus \{x + iy : x \leq 0, y = 0\}$$

onto $\mathbb{D}$? If yes, find one. If no, prove that there is none.

2. Let $f(z) = u(z) + iv(z)$ be a non-constant entire function.
a) Use the Poisson integral formula to show that for all $|z| < r$,

$$f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{it} + z}{re^{it} - z} u(re^{it}) dt.$$

b) Show that if $u(z) \leq |z|$ for all $z \in \mathbb{C}$, then f is linear.
c) Show that there are $a > 0, C > 0$ so that for all $r \geq 0$,

$$\max_{z \in \mathbb{C}, |z|=r} |e^{f(z)}| \geq Ce^{ar}.$$

3. Let

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^3}\right).$$

Show that for every $w \in \mathbb{C}$ the equation $f(z) = w$ has infinitely many solutions. You may use the conclusion of problem 2c), even if you did not solve that problem.

4. Let $\mathcal{F}$ be the family of all analytic functions $f : \mathbb{D} \to \mathbb{D}$.

a) State Montel's theorem and use it to show that there exists a function $F \in \mathcal{F}$ that maximizes $|f'(\frac{1}{2})|$. In other words, show that

$$\sup_{f \in \mathcal{F}} \left| f'\left(\frac{1}{2}\right) \right| = \left| F'\left(\frac{1}{2}\right) \right|$$

for some $F \in \mathcal{F}$.
b) Use Schwarz' Lemma to determine all functions F from part a).

5. Let $D \subsetneq \mathbb{C}$ be a simply connected domain. Determine a pair (a, b) of complex numbers such that the following statement is true: For every analytic function $f : D \to \mathbb{C} \setminus \{a, b\}$ there is an analytic function g on D with $f(z) = g(z)^3 + g(z)$ for all $z \in D$.

6. Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}.$$

7. Let $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda = \{a + b\tau \mid a, b \in \mathbb{Z}\}$. Let f be a non-constant meromorphic function with the property that

$$f(z + \omega) = f(z)$$

for all $z \in \mathbb{C}$ and $\omega \in \Lambda$ (such an f is called *elliptic* with respect to Λ). For $a \in \mathbb{C}$, denote

$$P_a = \{a + t + s\tau : 0 \leq t < 1, 0 \leq s < 1\}.$$

a) Use the argument principle to show that for each $a \in \mathbb{C}$, f assumes each value $c \in \mathbb{C} \cup \{\infty\}$ the same number of times in P_a, counting multiplicities. This number is called the *degree* of f.
b) Show that the degree of f is at least 2.

8. State a version of the reflection principle, and use it to prove the following: If u is a non-constant harmonic function in $\mathbb{C}$ that vanishes on two intersecting lines, then the angle between the lines is a rational multiple of π.