Complex Analysis Preliminary Exam
Autumn 2013

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

In all of these problems, \( \mathbb{C} \) will denote the complex plane and \( \mathbb{D} = \{ z : |z| < 1 \} \) will denote the open unit disk. A **domain** is an open connected subset of \( \mathbb{C} \).

1. Calculate the following integral:
   \[
   \int_0^{\infty} \frac{\cos x - 1}{x^2} \, dx.
   \]

2. Suppose \( f(z) \) is analytic on the unit disc \( \mathbb{D} \) and continuous on the closed unit disc \( \overline{\mathbb{D}} \). Assume that \( f(z) = 0 \) on an arc of the circle \( |z| = 1 \). Show that \( f(z) \equiv 0 \).

3. Let \( \mathcal{F} \) be the class of holomorphic functions \( f \) in the slit half-plane
   \[
   S = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \setminus (0, 1]
   \]
satisfying \(|f(z)| < 1\) for all \( z \in S \) and \( f(\sqrt{2}) = 0 \). Find \( f \in \mathcal{F} \) such that \(|f(4)| = \sup_{g \in \mathcal{F}} |g(4)|\), and prove your answer correct. (You may write your function as a composition of several other functions, if that is more convenient.)

4. Suppose \( \{a_n\} \) is a sequence of distinct complex numbers with no limit point, and \( \{b_n\} \) is an arbitrary sequence of complex numbers. Prove that there exists an entire function \( f \) with \( f(a_n) = b_n \) for \( n = 1, 2, 3, \ldots \) (You may use general theorems such as the Mittag–Leffler or Weierstrass theorem without proof, as long as you provide their statements.)

5. Let \( f(z) \) be analytic in \( \mathbb{D} \) except for finitely many poles. Suppose that \( \lim_{z \to e^{i\theta}} |f(z)| = 1 \) for every \( \theta \in [0, 2\pi) \). Prove that \( f \) is a rational function.

6. Suppose \( f_n \) is a sequence of holomorphic functions in \( \mathbb{D} \) for which \( u(z) = \lim \text{Re}(f_n(z)) \) exists uniformly on every compact subset of \( \mathbb{D} \). Also assume there is a point \( z_0 \in \mathbb{D} \) for which \( v(z_0) = \lim \text{Im}(f_n(z_0)) \) exists. Prove \( f_n \) converges uniformly on every compact subset of \( \mathbb{D} \).

7. Suppose \( \phi : [-1, 1] \to \mathbb{C} \) is a continuous function, and define \( f : \mathbb{C} \setminus [-1, 1] \to \mathbb{C} \) by
   \[
   f(z) = \int_{-1}^{1} \frac{\phi(t)}{t-z} \, dt.
   \]
   Prove that \( f \) is holomorphic on \( \mathbb{C} \setminus [-1, 1] \), and find an expression for the Laurent coefficients of \( f \) about \( \infty \) in terms of \( \phi \).

8. For any positive integer \( n \), define \( f_n : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) by
   \[
   f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}.
   \]
   Let \( R > 0 \) be given. Show that for sufficiently large \( n \), all of the zeros of \( f_n \) lie inside the disk \( |z| < R \).