Complex Analysis Preliminary Exam

Autumn 2014

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Suppose $a > 0$ and $b > 0$. Compute

$$\int_0^\infty \frac{x\sin(ax)}{x^2+b^2}dx.$$

2. Let $f(z) = \dfrac{z-a}{1-\bar{a}z}$, where $|a| < 1$. Let $\mathbb{D} = \{z = x+iy : |z| < 1\}$. Prove that

$$\frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|dxdy = \frac{1-|a|^2}{|a|^2}\log\left(\frac{1}{1-|a|^2}\right).$$

3. Let W be an open set containing the real axis $\mathbb{R}$ in $\mathbb{C}$. Suppose f is analytic in W and

$$\mathrm{Im}(z)\mathrm{Im}(f(z)) \geq 0$$

for $z \in W$. Prove that

$$f'(z) > 0$$

for all $z \in \mathbb{R}$.

4. Suppose u is harmonic and bounded in a bounded region Ω. Suppose further that there is a $\zeta \in \partial\Omega$ and a neighborhood W of $\partial\Omega \setminus \{\zeta\}$ such that $u \leq 1$ on $W \cap \Omega$.

a. Prove that $u \leq 1$ on Ω.

b. Give an example to show that the result can fail if we do not assume u is bounded.
Hint: consider $v = u + \varepsilon \log|(z-\zeta)/R|$ where $\varepsilon > 0$ and R is the diameter of Ω.

5. Suppose u and v are positive harmonic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$, which are continuous on the closure of $\mathbb{D}$ and equal to 0 on an open arc $J \subset \partial\mathbb{D}$, with $1 \in J$. Prove

$$\lim_{z\in\mathbb{D}\to 1} \frac{u(z)}{v(z)}$$

exists and is positive.

6. Let $f(z)$ be a nowhere zero holomorphic function on $\mathbb{S} = \{z : 0 < \mathrm{Re} z < 1\}$ which is bounded uniformly, i.e. $|f(z)| \leq M < \infty$ for all $z \in \mathbb{S}$. Suppose that

$$\lim_{n\to\infty} f(\frac{1}{2} + ni) = 0$$

for $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} f(z+ni) = 0$$

for each $z \in \mathbb{S}$.

7. Let $B(z,r) = \{w : |w - z| < r\}$ be the open ball centered at z of radius r. Prove that there exists an entire function f with the property that for all $\varepsilon > 0$ there exists an $N < \infty$ (depending on ε) such that

$$|f(z) - \sin z| < \varepsilon \qquad on \bigcup_{n=N}^{\infty} B(2n, 1/3)$$

and

$$|f(z) - \cos z| < \varepsilon \qquad on \bigcup_{n=N}^{\infty} B(2n+1, 1/3).$$

8. Suppose g_1 and g_2 are entire with no common zeros. Show that there exist entire functions f_1 and f_2 such that

$$f_1 g_1 + f_2 g_2 = e^z.$$

Hint: $f_2 = \dfrac{e^z - f_1 g_1}{g_2}$. Try to find f_1 entire so that the right hand side is entire.