Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Suppose $a > 0$ and $b > 0$. Compute
\[ \int_{0}^{\infty} \frac{x \sin(ax)}{x^2 + b^2} \, dx. \]

2. Let $f(z) = \frac{z - a}{1 - \bar{a}z}$, where $|a| < 1$. Let $D = \{ z = x + iy : |z| < 1 \}$. Prove that
\[ \frac{1}{\pi} \int_{D} |f'(z)| \, dx \, dy = \frac{1 - |a|^2}{|a|^2} \log \left( \frac{1}{1 - |a|^2} \right). \]

3. Let $W$ be an open set containing the real axis $\mathbb{R}$ in $\mathbb{C}$. Suppose $f$ is analytic in $W$ and
\[ \text{Im}(z) \text{Im}(f(z)) \geq 0 \]
for $z \in W$. Prove that
\[ f'(z) > 0 \]
for all $z \in \mathbb{R}$.

4. Suppose $u$ is harmonic and bounded in a bounded region $\Omega$. Suppose further that there is a $\zeta \in \partial \Omega$ and a neighborhood $W$ of $\partial \Omega \setminus \{\zeta\}$ such that $u \leq 1$ on $W \cap \Omega$.
   a. Prove that $u \leq 1$ on $\Omega$.
   b. Give an example to show that the result can fail if we do not assume $u$ is bounded.
      Hint: consider $v = u + \varepsilon \log |(z - \zeta)/R|$ where $\varepsilon > 0$ and $R$ is the diameter of $\Omega$.

5. Suppose $u$ and $v$ are positive harmonic functions on the unit disk $D = \{ z : |z| < 1 \}$, which are continuous on the closure of $D$ and equal to 0 on an open arc $J \subset \partial D$, with $1 \in J$. Prove
\[ \lim_{z \to D^{-1}} \frac{u(z)}{v(z)} \]
extists and is positive.

6. Let $f(z)$ be a nowhere zero holomorphic function on $S = \{ z : 0 < \text{Re} z < 1 \}$ which is bounded uniformly, i.e. $|f(z)| \leq M < \infty$ for all $z \in S$. Suppose that
\[ \lim_{n \to \infty} f\left( \frac{1}{2} + ni \right) = 0 \]
for $n \in \mathbb{N}$. Prove that
\[ \lim_{n \to \infty} f(z + ni) = 0 \]
for each $z \in S$. 

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7. Let $B(z, r) = \{ w : |w - z| < r \}$ be the open ball centered at $z$ of radius $r$. Prove that there exists an entire function $f$ with the property that for all $\varepsilon > 0$ there exists an $N < \infty$ (depending on $\varepsilon$) such that

$$ |f(z) - \sin z| < \varepsilon \quad \text{on} \quad \bigcup_{n=N}^{\infty} B(2n, 1/3) $$

and

$$ |f(z) - \cos z| < \varepsilon \quad \text{on} \quad \bigcup_{n=N}^{\infty} B(2n + 1, 1/3). $$

8. Suppose $g_1$ and $g_2$ are entire with no common zeros. Show that there exist entire functions $f_1$ and $f_2$ such that

$$ f_1 g_1 + f_2 g_2 = e^z. $$

Hint: $f_2 = \frac{e^z - f_1 g_1}{g_2}$. Try to find $f_1$ entire so that the right hand side is entire.