Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

In all of these problems, \( \mathbb{C} \) denotes the complex plane and \( \mathbb{D} = \{ z : |z| < 1 \} \) denotes the open unit disk. A **domain** is an open connected subset of \( \mathbb{C} \).

1. Evaluate both integrals
   \[
   \int_0^\infty \frac{\log x}{1 + x^2} \, dx \quad \text{and} \quad \int_0^\infty \frac{(\log x)^2}{1 + x^2} \, dx.
   \]

2. State a definition of local uniform convergence of infinite products \( \prod_{n=0}^\infty f_n(z) \) of holomorphic functions \( f_n \) on a domain \( G \). Next, let \( p(z) = z^2 + z + 1 \) and show that the infinite product
   \[
   P(z) = \prod_{n=0}^\infty p(z^{3^n})
   \]
   converges locally uniformly in \( \mathbb{D} \). Finally, prove that \( P(z) = \frac{1}{1-z} \) for all \( |z| < 1 \).

3. How many roots does \( 2z^7 - 4z^3 + 1 = 0 \) have in the annulus \( 1 < |z| < 2 \)?

4. Suppose \( G \subset \mathbb{C} \) is a domain, and \( f_n : G \to \mathbb{C} \) is a sequence of holomorphic functions satisfying \( \int_{G} |f_n(x+iy)|^2 \, dx \, dy = 1 \) for each \( n \). Show that some subsequence converges to a holomorphic function on \( G \).

5. Suppose \( f \) is a nowhere vanishing holomorphic function on the punctured disk \( \mathbb{D} \setminus \{0\} \). Prove that there is an integer \( n \) and a holomorphic function \( g \) on \( \mathbb{D} \setminus \{0\} \) such that \( f(z) = z^n e^{g(z)} \).

6. Consider the domain \( G = \{ z \in \mathbb{C} : \text{Im} z > 0 \text{ and } |z-i| > 1/2 \} \). Determine all functions \( u : G \to \mathbb{R} \) that are bounded and harmonic on \( G \) and satisfy \( u(x) = 1 \) for \( x \in \mathbb{R} \) and \( u(z) = 2 \) for \( \{|z-i| = 1/2\} \). Hint: Construct a conformal map from \( G \) onto a concentric annulus.

7. Find a conformal map from \( \mathbb{D} \) onto the portion of the upper half plane inside \( \mathbb{D} \) and outside the circle \( |z - 1/2| = 1/2 \), \( G = \{ z \in \mathbb{D} : \text{Re} z > 0, |z - 1/2| > 1/2 \} \). You may express your map as a composition of simpler maps.

8. If \( f \) is holomorphic and bounded in \( \mathbb{C} \setminus \mathbb{D} \), and if \( f \) is real-valued on the vertical line segment \( \{-2 + iy : 0 \leq y \leq 1\} \), show that \( f \) is constant.