Complex Analysis Preliminary Exam

Autumn 2015

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

In all of these problems, $\mathbb{C}$ denotes the complex plane and $\mathbb{D} = \{z : |z| < 1\}$ denotes the open unit disk. A ***domain*** is an open connected subset of $\mathbb{C}$.

1. Evaluate both integrals
$$\int_0^\infty \frac{\log x}{1+x^2}\,dx \quad \text{and} \quad \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx.$$

2. State a definition of local uniform convergence of infinite products $\prod_{n=0}^\infty f_n(z)$ of holomorphic functions f_n on a domain G. Next, let $p(z) = z^2 + z + 1$ and show that the infinite product
$$P(z) = \prod_{n=0}^\infty p(z^{3^n})$$
converges locally uniformly in $\mathbb{D}$. Finally, prove that $P(z) = \frac{1}{1-z}$ for all $|z| < 1$.

3. How many roots does $2z^7 - 4z^3 + 1 = 0$ have in the annulus $1 < |z| < 2$?

4. Suppose $G \subset \mathbb{C}$ is a domain, and $f_n \colon G \to \mathbb{C}$ is a sequence of holomorphic functions satisfying $\iint_G |f_n(x+iy)|^2 dx\,dy = 1$ for each n. Show that some subsequence converges to a holomorphic function on G.

5. Suppose f is a nowhere vanishing holomorphic function on the punctured disk $\mathbb{D} \setminus \{0\}$. Prove that there is an integer n and a holomorphic function g on $\mathbb{D} \setminus \{0\}$ such that $f(z) = z^n e^{g(z)}$.

6. Consider the domain $G = \{z \in \mathbb{C} : \text{Im} z > 0 \text{ and } |z - i| > 1/2\}$. Determine all functions $u : G \to \mathbb{R}$ that are bounded and harmonic on G and satisfy $u(x) = 1$ for $x \in \mathbb{R}$ and $u(z) = 2$ for $\{|z - i| = 1/2\}$. Hint: Construct a conformal map from G onto a concentric annulus.

7. Find a conformal map from $\mathbb{D}$ onto the portion of the upper half plane inside $\mathbb{D}$ and outside the circle $|z - 1/2| = 1/2$, $G = \{z \in \mathbb{D} : \text{Re} z > 0, |z - 1/2| > 1/2\}$. You may express your map as a composition of simpler maps.

8. If f is holomorphic and bounded in $\mathbb{C} \setminus \mathbb{D}$, and if f is real-valued on the vertical line segment $\{-2 + iy : 0 \le y \le 1\}$, show that f is constant.