Complex Analysis Preliminary Exam

Autumn 2016

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. By means of the calculus of residues, evaluate

$$\int_0^\infty \frac{\sqrt{x}\log x}{(1+x)^2}dx.$$

Justify all limiting operations and estimates.

2. Find all functions f analytic in the interior of the rectangle with vertices $1+i$, $1-i/2$, $-1/2-i/2$, $-1/2+i$ such that $f(z+1)=f(z)$ and $f(z+i)=1+f(z)$ wherever these equalities are defined.

3. Suppose P is a polynomial of degree $n>1$. Let

$$E=\{z\in\mathbb{C}:P^{-1}(z) \text{ consists of } n \text{ distinct points }\}.$$

Let Q be another polynomial and define

$$\widetilde{Q}:E\to\mathbb{C}$$

by

$$\widetilde{Q}(z)=\sum_{\zeta\in P^{-1}(z)}Q(\zeta).$$

(a) Prove E is open.
(b) Prove $\widetilde{Q}$ extends to be an entire function.
(c) Prove that the extension of $\widetilde{Q}$ is a polynomial.

4. Let p and q be distinct points in the boundary of the unit disk $\mathbb{D}=\{z:|z|<1\}$, and let U_p and U_q be neighborhoods of p and q respectively in $\mathbb{C}$. Prove that no matter how small U_p and U_q may be, there is a conformal map ϕ of $\mathbb{D}$ onto $\mathbb{D}$ so that $\phi(\mathbb{D}\setminus U_q)\subset U_p$.

5. Let $f(t)$ be a real valued infinitely differentiable function on $[0,1]$. Define $\Omega=\{z: \operatorname{Re}(z)>-1\}$ and $g(z):\Omega\to\mathbb{C}$ by

$$g(z)=\int_0^1 t^z f(t)dt.$$

(a) Prove that g is holomorphic on Ω.
(b) Prove that g can be continued analytically to a meromorphic function on the entire plane.

6. Let $S=\{z:|z|<1,\operatorname{Re}(z)>0\}$. Find (explicitly) a bounded harmonic function u defined on S such that $u(z)\to 0$ as $z\to it$, with $-1<t<1$ and $u(z)\to 1$ as $z\to a+ib$ with $a>0$, $-1<b<1$ and $a^2+b^2=1$. A complete answer will be in terms of elementary functions and not involve any integrals. Prove that u is unique.

7. Suppose f is a function defined on the unit disk $\mathbb{D}=\{z:|z|<1\}$ with the property that for each triple of points $a,b,c\in\mathbb{D}$ there is an analytic function g (possibly depending on a,b,c) which is bounded by 1 and satisfies $g(a)=f(a)$, $g(b)=f(b)$, and $g(c)=f(c)$. Prove that f is analytic in $\mathbb{D}$ and is bounded by 1. Hint:

Consider first the special case where $a = f(a) = 0$. If z_n is a sequence tending to 0, prove that $f(z_n)/z_n$ is a Cauchy sequence.

8. (a) Suppose $0 < a_n < 1$. Prove that if

$$\lim_{M\to\infty} \prod_1^M (1 - a_n) = 0,$$

then $\sum a_n$ diverges. Don't just quote a theorem or lemma, prove it directly.

(b) Let $p_k(z)$ be a polynomial of degree k with $p_k(0) = 1$ such that p_k has no zeros in $\overline{\mathbb{D}(0, k^3)}$. Show that $\prod_1^\infty p_k(z)$ converges locally uniformly in $\mathbb{C}$.