LINEAR ANALYSIS PRELIM EXAM

Autumn 2005

- There are 8 questions. You are guaranteed to pass the exam if you answer at least **four** questions correctly. Partial answers may count, but in general it is preferable to give complete answers to fewer questions than partial answers to more questions.
- If you cannot answer a part of a question, you may assume the result and proceed to a subsequent part.

1. Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric $n \times n$ matrix. Let $\{v_1, v_2, \dots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ (using the usual inner product on $\mathbb{R}^n$) consisting of eigenvectors of A, corresponding to the eigenvalues λ_1, λ_2, ..., λ_n of A.

   (a) Show that the set $\left\{v_k\, v_\ell^T \in \mathbb{R}^{n\times n} \ : 1 \leq k \leq n, 1 \leq \ell \leq n\right\}$ forms a basis of $\mathbb{R}^{n\times n}$.

   (b) Show that

   $$A = \sum_{k=1}^{n} \lambda_k\, v_k\, v_k^T$$

   is the representation of A as a linear combination of the basis of $\mathbb{R}^{n\times n}$ in part (a).

   (c) Define the linear map $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

   $$\Phi(M) = AM - MA.$$

   The null space of Φ is clearly the subspace of $\mathbb{R}^{n\times n}$ consisting of all matrices $M \in \mathbb{R}^{n\times n}$ that commute with A, i.e., for which $AM = MA$. Show that the dimension of the null space of Φ is

   $$\sum_{\lambda \in \sigma(A)} m_\lambda^2,$$

   i.e., the sum of the squares of the multiplicities of the distinct eigenvalues of A. (Hint: Express M in terms of the basis of $\mathbb{R}^{n\times n}$ in part (a).)

2. Find all distributions $u \in \mathcal{D}'(\mathbb{R}^2)$ of order at most 2 for which $(x^2 + y^2)u = 0$.

3. Suppose $\alpha(t)$ is a continuous complex-valued function of $t \geq 0$. Let $r(t)$ be a solution of the initial value problem

   $$\begin{aligned} \frac{dr}{dt} &= ir + \alpha - \overline{\alpha} r^2 \qquad (t \geq 0), \\ r(0) &= 0, \end{aligned}$$

   and assume that $r(t)$ is defined for all $t \geq 0$.

   (a) Derive an expression for $\frac{d}{dt}\left(|r|^2\right) = r\frac{d\overline{r}}{dt} + \frac{dr}{dt}\overline{r}$, and show that whenever $|r(t)| \leq 1$,

   $$\frac{d}{dt}\left(1 - |r(t)|^2\right) \geq -2\,|\alpha(t)|\,|r(t)|\left(1 - |r(t)|^2\right).$$

   (b) Suppose in addition that

   $$\int_0^\infty |\alpha(t)| dt < \infty.$$

   Show that $|r(t)| < 1$ for all $t > 0$.

4. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a continuous matrix-valued function that is periodic of period $p > 0$, i.e., $A(t+p) = A(t)$. Let $Y : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a matrix-valued solution of the initial value problem

$$\begin{aligned} Y'(t) &= A(t)Y(t) \qquad (t \in \mathbb{R}), \\ Y(0) &= C, \end{aligned}$$

where $C \in \mathbb{R}^{n\times n}$ is a given constant matrix.

(a) Show that if C is invertible, then $Y(t)$ is also invertible for all $t \in \mathbb{R}$.

(b) Suppose C is invertible. Show that the matrix

$$\Omega = Y(t)^{-1}Y(t+p)$$

is independent of t.

(c) Show that the eigenvalues of Ω are independent of the choice of the invertible matrix C.

5. Let $B_1 = \{(x,y) : x_1^2 + x_2^2 \leq 1\}$ be the unit ball in $\mathbb{R}^2$, and $S^1 = \{(\xi_1, \xi_2) : \xi_1^2 + \xi_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Define the operator $H : L^2(B_1) \to L^2(S^1)$ by

$$Hf = \widehat{f}(\xi)\Big|_{|\xi|=1},$$

i.e., Hf is the restriction of the Fourier transform of f to the unit circle S^1.

(a) Show that $L^2(B_1) \subset L^1(\mathbb{R}^2)$, and conclude that the restriction of

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx$$

to S^1 is well-defined for $f \in L^2(B_1)$.

(b) Using the correspondence $(\cos\theta, \sin\theta) \leftrightarrow \theta$ to identify S^1 with the interval $(-\pi, \pi]$, express H as an integral operator, i.e., find its kernel $h(\theta, x)$ such that

$$[Hf]\,(\theta) = \int_{|x|\leq 1} h(\theta, x) f(x)\, dx \quad (-\pi < \theta \leq \pi).$$

(c) Show that $H : L^2(B_1) \to L^2(S^1)$ is a compact operator.

(d) Using the inner product $(f,g) = \int_{|x|\leq 1} f(x)\overline{g(x)}\, dx$ on $L^2(B_1)$ and the inner product $(\phi, \psi) = \int_{-\pi}^{\pi} \phi(\theta)\overline{\psi(\theta)}\, d\theta$ on $L^2(S^1)$, express the Hilbert-space adjoint H^* of H

$$H^* : L^2(S^1) \to L^2(B_1)$$

as an integral operator, i.e., find its kernel $k(x, \theta)$ such that

$$[H^*\phi]\,(x) = \int_{-\pi}^{\pi} k(x,\theta)\phi(\theta)\, d\theta \quad (|x| \leq 1).$$

6. Let $q \in L^1(\mathbb{R})$. Define the operator T by $[Tf](x) = \int_{\mathbb{R}} q(x-y)f(y)\,dy$.

(a) Show that T maps $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$.

(b) Show that $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is not a compact operator unless $q = 0$ a.e.

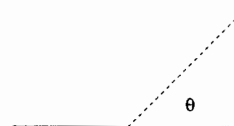
7. Let V_1 and V_2 be subspaces of $\mathbb{R}^n$ which satisfy $\mathbb{R}^n = V_1 \oplus V_2$, but are not orthogonal (in the usual inner product on $\mathbb{R}^n$). Let Q denote the projection of $\mathbb{R}^n$ onto V_1 along V_2. Let P_1 be the *orthogonal* projection of $\mathbb{R}^n$ onto V_1 (caution: $P_1 \neq Q$), and let P_2 be the *orthogonal* projection of $\mathbb{R}^n$ onto V_2.

(a) Show that $I - P_1P_2$ is invertible. (Hint: Show that $x = P_1P_2x$ implies that $x = 0$.)

(b) Show that $P_1(I - P_2) = P_1(I - P_1P_2)$.

(c) Show that $Q = (I - P_1P_2)^{-1}P_1(I - P_2)$.

8. In $\mathbb{R}^2 \setminus \{(0,0)\}$, let θ denote the branch of the polar-coordinate angle (i.e., $\tan\theta = y/x$) for which $-\pi < \theta \leq \pi$.



Clearly $\theta(x,y) \in L^1_{loc}(\mathbb{R}^2)$, so we can view θ as a distribution on $\mathbb{R}^2$. Show that the Laplacian $\Delta\theta \equiv \dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2}$ of θ (in the sense of distributions) satisfies

$$\Delta\theta = 2\pi\frac{\partial v}{\partial y},$$

where $v \in \mathcal{D}'(\mathbb{R}^2)$ is the distribution

$$\langle v, \phi\rangle = \int_{-\infty}^{0} \phi(x,0)\,dx \qquad (\phi \in \mathcal{D}(\mathbb{R}^2)).$$