LINEAR ANALYSIS PRELIM EXAM

Autumn 2005

- There are 8 questions. You are guaranteed to pass the exam if you answer at least four questions correctly. Partial answers may count, but in general it is preferable to give complete answers to fewer questions than partial answers to more questions.

- If you cannot answer a part of a question, you may assume the result and proceed to a subsequent part.
1. Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric $n \times n$ matrix. Let \( \{v_1, v_2, \ldots, v_n\} \) be an orthonormal basis of $\mathbb{R}^n$ (using the usual inner product on $\mathbb{R}^n$) consisting of eigenvectors of $A$, corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A$.

(a) Show that the set \( \{v_k v_k^T : 1 \leq k \leq n, 1 \leq \ell \leq n\} \) forms a basis of $\mathbb{R}^{n \times n}$.

(b) Show that
\[
A = \sum_{k=1}^{n} \lambda_k v_k v_k^T
\]
is the representation of $A$ as a linear combination of the basis of $\mathbb{R}^{n \times n}$ in part (a).

(c) Define the linear map $\Phi : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ by
\[
\Phi(M) = AM - MA.
\]
The null space of $\Phi$ is clearly the subspace of $\mathbb{R}^{n \times n}$ consisting of all matrices $M \in \mathbb{R}^{n \times n}$ that commute with $A$, i.e., for which $AM = MA$. Show that the dimension of the null space of $\Phi$ is
\[
\sum_{\lambda \in \sigma(A)} m_{\lambda}^2,
\]
i.e., the sum of the squares of the multiplicities of the distinct eigenvalues of $A$.

(Hint: Express $M$ in terms of the basis of $\mathbb{R}^{n \times n}$ in part (a).)

2. Find all distributions $u \in \mathcal{D}'(\mathbb{R}^2)$ of order at most 2 for which $(x^2 + y^2)u = 0$.

3. Suppose $\alpha(t)$ is a continuous complex-valued function of $t \geq 0$. Let $r(t)$ be a solution of the initial value problem
\[
\frac{dr}{dt} = ir + \alpha - \overline{\alpha}r^2 \quad (t \geq 0),
\]
r(0) = 0,
and assume that $r(t)$ is defined for all $t \geq 0$.

(a) Derive an expression for $\frac{d}{dt} (|r|^2) = r \frac{dr}{dt} + \overline{r} \frac{dr}{dt}$, and show that whenever $|r(t)| \leq 1,
\[
\frac{d}{dt} (1 - |r(t)|^2) \geq -2 \Re \alpha(t)|r(t)| (1 - |r(t)|^2).
\]

(b) Suppose in addition that
\[
\int_0^\infty |\alpha(t)| dt < \infty.
\]
Show that $|r(t)| < 1$ for all $t > 5$. 

1
4. Let $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ be a continuous matrix-valued function that is periodic of period $p > 0$, i.e., $A(t + p) = A(t)$. Let $Y : \mathbb{R} \to \mathbb{R}^{n \times n}$ be a matrix-valued solution of the initial value problem
\[
Y'(t) = A(t)Y(t) \quad (t \in \mathbb{R}),
\]
\[
Y(0) = C,
\]
where $C \in \mathbb{R}^{n \times n}$ is a given constant matrix.

(a) Show that if $C$ is invertible, then $Y(t)$ is also invertible for all $t \in \mathbb{R}$.
(b) Suppose $C$ is invertible. Show that the matrix
\[
\Omega = Y(t)^{-1}Y(t + p)
\]

is independent of $t$.
(c) Show that the eigenvalues of $\Omega$ are independent of the choice of the invertible matrix $C$.

5. Let $B_1 = \{(x, y) : x^2 + y^2 \leq 1\}$ be the unit ball in $\mathbb{R}^2$, and $S^1 = \{\xi_1, \xi_2 : \xi_1^2 + \xi_2^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Define the operator $H : L^2(B_1) \to L^2(S^1)$ by
\[
Hf = \left. \tilde{f}(\xi) \right|_{||\xi||=1},
\]
i.e., $Hf$ is the restriction of the Fourier transform of $f$ to the unit circle $S^1$.

(a) Show that $L^2(B_1) \subset L^1(\mathbb{R}^2)$, and conclude that the restriction of
\[
\tilde{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x)dx
\]
to $S^1$ is well-defined for $f \in L^2(B_1)$.
(b) Using the correspondence $(\cos \theta, \sin \theta) \leftrightarrow \theta$ to identify $S^1$ with the interval $(-\pi, \pi]$, express $H$ as an integral operator, i.e., find its kernel $h(\theta, x)$ such that
\[
[Hf](\theta) = \int_{\mathbb{R}^2} h(\theta, x)f(x)dx \quad (-\pi < \theta \leq \pi).
\]
(c) Show that $H : L^2(B_1) \to L^2(S^1)$ is a compact operator.
(d) Using the inner product $(f, g) = \int_{B_1} f(x)g(x)dx$ on $L^2(B_1)$ and the inner product $(\phi, \psi) = \int_{-\pi}^{\pi} \phi(\theta)\overline{\psi(\theta)}d\theta$ on $L^2(S^1)$, express the Hilbert-space adjoint $H^*$ of $H$
\[
H^* : L^2(S^1) \to L^2(B_1)
\]
as an integral operator, i.e., find its kernel $k(x, \theta)$ such that
\[
[H^*\phi](x) = \int_{-\pi}^{\pi} k(x, \theta)\phi(\theta)d\theta \quad (||x|| \leq 1).
\]
6. Let \( q \in L^1(\mathbb{R}) \). Define the operator \( T \) by \( [Tf](x) = \int_{\mathbb{R}} q(x - y) f(y) \, dy \).

(a) Show that \( T \) maps \( L^2(\mathbb{R}) \) into \( L^2(\mathbb{R}) \).
(b) Show that \( T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is not a compact operator unless \( q = 0 \) a.e.

7. Let \( V_1 \) and \( V_2 \) be subspaces of \( \mathbb{R}^{n} \) which satisfy \( \mathbb{R}^{n} = V_1 \oplus V_2 \), but are not orthogonal (in the usual inner product on \( \mathbb{R}^{n} \)). Let \( Q \) denote the projection of \( \mathbb{R}^{n} \) onto \( V_1 \) along \( V_2 \). Let \( P_1 \) be the orthogonal projection of \( \mathbb{R}^{n} \) onto \( V_1 \) (caution: \( P_1 \neq Q \)), and let \( P_2 \) be the orthogonal projection of \( \mathbb{R}^{n} \) onto \( V_2 \).

(a) Show that \( I - P_1 P_2 \) is invertible. (Hint: Show that \( x = P_1 P_2 x \) implies that \( x = 0 \).)
(b) Show that \( P_1 (I - P_2) = P_1 (I - P_1 P_2) \).
(c) Show that \( Q = (I - P_1 P_2)^{-1} P_1 (I - P_2) \).

8. In \( \mathbb{R}^{n} \setminus \{(0,0)\} \), let \( \theta \) denote the branch of the polar-coordinate angle (i.e., \( \tan \theta = y/x \)) for which \( -\pi < \theta \leq \pi \).

Clearly \( \theta(x,y) \in L^1_{\text{loc}}(\mathbb{R}^2) \), so we can view \( \theta \) as a distribution on \( \mathbb{R}^{2} \). Show that the Laplacian \( \Delta \theta = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \) of \( \theta \) (in the sense of distributions) satisfies

\[
\Delta \theta = 2\pi \frac{\partial v}{\partial y},
\]

where \( v \in \mathcal{D}'(\mathbb{R}^2) \) is the distribution

\[
\langle v, \phi \rangle = \int_{-\infty}^{0} \phi(x,0) \, dx \quad (\phi \in \mathcal{D}(\mathbb{R}^2)).
\]