There are 8 questions. You are guaranteed to pass the exam if you give complete, correct answers to at least four of the questions. Partial answers may count but, in general, it is preferable to give complete answers to fewer questions rather than partial answers to more questions.

If you cannot answer a part of a question, you may assume the result and proceed to a subsequent part.
1. (a) What are the possible Jordan forms of a matrix \( A \in \mathbb{C}^{4 \times 4} \) satisfying \((A - I)^2 = 0\)? (Ignore different permutations of the Jordan blocks.)

(b) Let \( J_n \) be an \( n \) by \( n \) Jordan block with eigenvalue 0. What is the Jordan form of \( J_n^2 \)?

2. Show that every \( n \) by \( n \) matrix \( A \) is unitarily similar to a matrix with equal diagonal entries; i.e., there exists an \( n \) by \( n \) matrix \( U \) with \( U^*U = UU^* = I \) (the identity) such that \( U^*AU \) has all of its diagonal entries equal to \( \text{tr}(A)/n \) (where \( \text{tr}(\cdot) \) denotes the trace). [Hint: One approach is to use induction on the size \( n \) of the matrix, together with showing that there is a unit vector \( u_1 \) such that \( u_1^*Au_1 = \text{tr}(A)/n \).]

3. Show that there exist infinitely many solutions to the ODE \( u'(t) = \sqrt{u(t)}, t > 0, \) with \( u(0) = 0 \).

4. Consider the initial value ODE problem \( u'(t) = f(t, u), 0 \leq t \leq T, u(0) = u_0 \), where \( f \) is \( C^\infty \) in \( t \) and \( u \). Consider numerical methods of the form

\[
 u_{i+2} + a_1 u_{i+1} + a_0 u_i = hbf(t_{i+2}, u_{i+2}),
\]

where \( u_i \) represents the approximate solution at \( t_i = ih \), \( h = T/N \).

(a) Determine the coefficients \( a_0, a_1, \) and \( b \) that give the highest order local truncation error for the method, and show what that order is.

(b) Is the resulting method convergent (i.e., does the approximate solution converge uniformly to the true solution on the mesh points as \( h \to 0 \))? Explain why or why not (i.e., either prove your answer directly or quote a theorem and show that all of the hypotheses of the theorem are satisfied).

5. Let \( X \) be a Banach space and \( T \) a bounded linear operator on \( X \). The numerical range of \( T \) is the subset of \( \mathbb{C} \) defined by

\[
 W(T) = \{ f(Tx) : x \in X, f \in X^*, \|x\| = \|f\| = f(x) = 1 \}.
\]

Show that the spectrum of \( T \) is a subset of the closure of \( W(T) \): \( \sigma(T) \subset \overline{W(T)} \).

6. Solve the Cauchy problem

\[
 \begin{cases}
    u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } r > 0 \\
    u(r, \theta) = \sin^3 \theta, & u_r(r, \theta) = \cos 3\theta & \text{for } r = 1
\end{cases}
\]
7. Find all $C^2$ solutions to the heat equation

$$u_t - u_{xx} = t - x^2 \quad \text{in} \quad \mathbb{R}^2$$

satisfying

$$\lim_{|x|+|t| \to \infty} \frac{|u(x, t)|}{|x|^5 + |t|^5} = 0.$$

Justify your answer.

8. Find a constant coefficient partial differential operator $L = a\partial_x^2 + b\partial_x \partial_y + c\partial_y^2$ such that $L\chi = \delta$, where $\chi$ is the characteristic function of the domain \{(x, y) \in \mathbb{R}^2 \mid y \geq |x|\}.