LINEAR ANALYSIS PRELIM EXAM

Autumn 2008

• There are 8 questions. You are guaranteed to pass the exam if you give complete, correct answers to at least 4 of the questions.

• If you cannot answer a part of a question, you may assume the result and proceed to a subsequent part.

Notation
• $\mathbb{R}^{n \times m}$ denotes the $n \times m$ real matrices.
• $\mathbb{M}_n$ denotes the $n \times n$ complex matrices.
• $\text{GL}_n(\mathbb{C}) \subset \mathbb{M}_n$ denotes the subset of nonsingular matrices.
• $\mathbb{H}_n$ denotes the Hermitian symmetric matrices.
• $H \in \mathbb{H}_n$ is said to be positive definite ($H > 0$) if $x^T H x > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$.
• $H \in \mathbb{H}_n$ is said to be positive semi–definite ($H \geq 0$) if $x^T H x \geq 0$ for all $x \in \mathbb{C}^n$.
• $\mathbb{S}_n$ denotes the real symmetric matrices
• $H \in \mathbb{S}_n$ is said to be positive definite ($H > 0$) if $x^T H x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.
• $H \in \mathbb{S}_n$ is said to be positive semi–definite ($H \geq 0$) if $x^T H x \geq 0$ for all $x \in \mathbb{R}^n$.
• The singular values of a matrix $A$ are the eigenvalues of the matrix $\sqrt{A^* A}$.
• $\mathcal{S}'(\mathbb{R}^n)$ is the class of tempered distributions.
1. Given $A \in \mathbb{M}_n$, define the spectral abscissa of $A$, $\alpha(A)$, to be the maximum of the real parts of the spectrum of $A$,

$$
\alpha(A) = \max \{ \text{Re}\lambda \mid \det(A - \lambda I) = 0 \},
$$

where $\text{Re}(\lambda)$ is the real part of $\lambda$, and define the Hermitian part of $A$ to be the Hermitian symmetric matrix

$$
\mathcal{H}(A) = \frac{1}{2}(A + A^*) .
$$

(a) Show that $\alpha(A) \leq \alpha(\mathcal{H}(A))$.  \emph{Hint: Rayleigh Quotients may help.}
(b) For every $A \in \mathbb{M}_n$ and $\epsilon > 0$, show that there exists a unitary matrix $Q$ and a real diagonal matrix $T$ such that $\hat{A} = T^{-1}Q^*AQ$ is upper triangular with all off diagonal elements having magnitude less than $\epsilon$.
(c) Show that

$$
\alpha(A) = \inf_{X \in \text{GL}_n(\mathbb{C})} \alpha(\mathcal{H}(X^{-1}AX)).
$$

You may use, without proof, the fact that the spectrum depends continuously on the matrix elements.

2. Let $r, n$ and $m$ be positive integers satisfying $1 \leq r \leq \min\{n, m\}$.
(a) Let $\{v^1, \ldots, v^r\}$ and $\{w^1, \ldots, w^r\}$ be orthonormal sets of vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Show that

$$
\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} v^j \\ w^j \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -v^j \\ w^j \end{pmatrix} \mid j = 1, 2, \ldots, r \right\}
$$

is and orthonormal set of vectors in $\mathbb{C}^{n+m}$.
(b) Let $A \in \mathbb{M}_n$. Show that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the $n$ singular values of $A$ if and only if $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \geq -\sigma_n \geq \cdots \geq -\sigma_2 \geq -\sigma_1$ are the $2n$ eigenvalues of the symmetric matrix

$$
\begin{bmatrix}
0 & A \\
A^* & 0
\end{bmatrix} .
$$

3. Let $H \in \mathcal{S}_n$, $u \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Consider the block matrix

$$
\hat{H} := \begin{bmatrix}
H & u \\
u^T & \alpha
\end{bmatrix} .
$$

(a) Show that $\hat{H}$ is positive definite if and only if $H$ is positive definite and $\alpha > u^T H^{-1} u$.
(b) Show that $\hat{H}$ is positive semi–definite if and only if $H$ is positive semi–definite and there exists a vector $z \in \mathbb{R}^n$ such that $u = Hz$ and $\alpha \geq z^T Hz$. 

4. Let \( H \) be a real Hilbert space and \( b(u,v) \) a bilinear form satisfying:
\[
\sup_{\|v\|=1} \sup_{\|u\|=1} b(u,v) \leq C_0 \quad (a)
\]
\[
\inf_{\|v\|=1} \sup_{\|u\|=1} b(u,v) \geq C_1 \quad (b)
\]
\[
\inf_{\|u\|=1} \sup_{\|v\|=1} b(u,v) \geq C_2 \quad (c)
\]
where \( C_0, C_1, \) and \( C_2 \) are positive constants. Prove that there exists a unique bounded linear map \( B \), with bounded inverse, mapping \( H \) to itself, such that
\[
b(u,v) = (u, Bv)
\]
where \((f,g)\) denotes the Hilbert space inner product of \( f \) and \( g \).

5. Let \( A: \mathbb{R} \to \mathbb{R}^{n \times n} \), \( B: \mathbb{R} \to \mathbb{R}^{n \times m} \) and \( C: \mathbb{R} \to \mathbb{R}^{n \times k} \) be continuous matrix valued mappings and \( u: \mathbb{R} \to \mathbb{R}^m \) a continuous vector valued function. An input-output system \( S \) is an initial value problem
\[
\dot{x} = A(t)x + B(t)u
\]
\[
x(0) = 0
\]
together with an output relation
\[
v = C(t)x.
\]
The system \( S \) is said to be I-O stable if there is a constant \( K > 0 \) such that
\[
\sup_{t \geq 0} |v(t)| \leq K \sup_{t \geq 0} |u(t)|
\]
(a) Let \( \Phi \) be a fundamental matrix normalized at \( t_0 = 0 \) for the system \( \dot{x} = A(t)x \), and define \( h: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{k \times m} \) by
\[
h(t,s) = C(t)\Phi(t)\Phi^{-1}(s)B(s).
\]
Show that, if there exists \( M > 0 \) such that
\[
\int_0^t |h(t,s)| \, ds \leq M \quad \text{for all } \ t \geq 0.
\]
then the system \( S \) is I-O stable.

(b) Next suppose that \( A, B, \) and \( C \) are constant matrices. Show that, if all of the eigenvalues of \( A \) have negative real part, then the system \( S \) is I-O stable.

(c) Finally, assume that \( A, B, \) and \( C \) are constant and also that \( B \) and \( C \) are invertible. In this case, show that, the system \( S \) is I-O stable if and only if all of the eigenvalues of \( A \) have negative real part.
6. Let \( f(x) \) be a continuous bounded (i.e. there is a constant \( M \) such that \( |f(x)| \leq M \forall x \in \mathbb{R} \)) function on the real line. Prove that there is a unique bounded \( C^2 \) solution \( u \) of the differential equation
\[
\left( \frac{d}{dx} - 1 \right) \left( \frac{d}{dx} + 2 \right) u = f
\]
Writing an explicit formula for \( u \) as an integral or a sum of integrals will help to show that \( u \) is bounded.

7. Let \( q(x) \) be a continuous real-valued function on the interval \([0, 1]\). Consider the Sturm-Liouville problem
\[
u'' + q(x)u = \lambda u
\]
\[
u'(0) = u(0) \quad u(1) = 0
\]
Show that all eigenvalues are real, simple (i.e. if two eigenfunctions have the same eigenvalue, then one is a constant multiple of the other), and satisfy
\[
\lambda \leq \max_{x \in [0,1]} q(x)
\]
*Hint: Prove and use the inequality \( |u(0)|^2 \leq \int_0^1 |u'(t)|^2 dt \)*

8. a) Show that, for any tempered distribution \( \mu \in S'(\mathbb{R}^n) \), there is a unique tempered distribution \( \omega \in S'(\mathbb{R}^n) \) that solves
\[
\Delta \omega - \omega = \mu
\]
b) If \( \mu \) is the distribution
\[
\langle \mu, \phi \rangle = \int_{-\infty}^{\infty} \phi(0, y)dy
\]
show that \( \omega \) is a function, defined at every point \((x, y) \in \mathbb{R}^2\) and bounded by a constant.
c) For each \( 0 \leq \theta < 2\pi \), calculate
\[
\lim_{r \to \infty} \omega(r \cos(\theta), r \sin(\theta))
\]