LINEAR ANALYSIS PRELIM EXAM

Autumn 2008

- There are 8 questions. You are guaranteed to pass the exam if you give complete, correct answers to at least 4 of the questions.
- If you cannot answer a part of a question, you may assume the result and proceed to a subsequent part.

<u>Notation</u>

- $\mathbb{R}^{n\times m}$ denotes the $n \times m$ real matrices.
- $\mathbb{M}_n$ denotes the $n \times n$ complex matrices.
- $\mathrm{GL}_n(\mathbb{C}) \subset \mathbb{M}_n$ denotes the subset of nonsingular matrices.
- $\mathbb{H}_n$ denotes the Hermitian symmetric matrices.
- $H \in \mathbb{H}_n$ is said to be *positive definite* $(H > 0)$ if $x^T Hx > 0$ for all $x \in \mathbb{C}^n$ with $x \neq 0$.
- $H \in \mathbb{H}_n$ is said to be *positive semi–definite* $(H \geq 0)$ if $x^T Hx \geq 0$ for all $x \in \mathbb{C}^n$.
- $\mathbb{S}_n$ denotes the real symmetric matrices
- $H \in \mathbb{S}_n$ is said to be *positive definite* $(H > 0)$ if $x^T Hx > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.
- $H \in \mathbb{S}_n$ is said to be *positive semi–definite* $(H \geq 0)$ if $x^T Hx \geq 0$ for all $x \in \mathbb{R}^n$.
- The singular values of a matrix A are the eigenvalues of the matrix $\sqrt{A^*A}$.
- $S'(\mathbb{R}^n)$ is the class of tempered distributions.

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1. Given $A \in \mathbb{M}_n$, define the spectral abscissa of A, $\alpha(A)$, to be the maximum of the real parts of the spectrum of A,

$$\alpha(A) = \max\{\mathsf{Re}\lambda \mid \det(A - \lambda I) = 0\},$$

where $\mathsf{Re}(\lambda)$ is the real part of λ, and define the Hermitian part of A to be the Hermitian symmetric matrix

$$\mathcal{H}(A) = \frac{1}{2}(A + A^*) .$$

(a) Show that $\alpha(A) \le \alpha(\mathcal{H}(A))$. *Hint: Rayleigh Quotients may help.*

(b) For every $A \in \mathbb{M}_n$ and $\epsilon > 0$, show that there exists a unitary matrix Q and a real diagonal matrix T such that $\hat{A} = T^{-1}Q^*AQT$ is upper triangular with all off diagonal elements having magnitude less than ϵ.

(c) Show that

$$\alpha(A) = \inf_{X \in \mathrm{GL}_n(\mathbb{C})} \alpha(\mathcal{H}(X^{-1}AX)).$$

You may use, without proof, the fact that the spectrum depends continuously on the matrix elements.

2. Let r, n and m be positive integers satisfying $1 \le r \le \min\{n, m\}$.

(a) Let $\{v^1, \ldots, v^r\}$ and $\{w^1, \ldots, w^r\}$ be orthonormal sets of vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Show that

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} v^j \\ w^j \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -v^j \\ w^j \end{pmatrix} \mid j = 1, 2, \ldots, r \right\}$$

is and orthonormal set of vectors in $\mathbb{C}^{n+m}$.

(b) Let $A \in \mathbb{M}_n$. Show that $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$ are the n singular values of A if and only if $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0 \ge -\sigma_n \ge \cdots \ge -\sigma_2 \ge -\sigma_1$ are the $2n$ eigenvalues of the symmetric matrix

$$\begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} .$$

3. Let $H \in \mathcal{S}_n$, $u \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Consider the block matrix

$$\widehat{H} := \begin{bmatrix} H & u \\ u^T & \alpha \end{bmatrix} .$$

(a) Show that $\widehat{H}$ is positive definite if and only if H is positive definite and $\alpha > u^T H^{-1} u$.

(b) Show that $\widehat{H}$ is positive semi–definite if and only if H is positive semi–definite and there exists a vector $z \in \mathbb{R}^n$ such that $u = Hz$ and $\alpha \ge z^T H z$.

4. Let H be a real Hilbert space and $b(u,v)$ a bilinear form satisfying:

$$\sup_{||v||=1} \sup_{||u||=1} b(u,v) \le C_0 \qquad \text{(a)}$$

$$\inf_{||v||=1} \sup_{||u||=1} b(u,v) \ge C_1 \qquad \text{(b)}$$

$$\inf_{||u||=1} \sup_{||v||=1} b(u,v) \ge C_2 \qquad \text{(c)}$$

where C_0, C_1, and C_2 are positive constants. Prove that there exists a unique bounded linear map B, with bounded inverse, mapping H to itself, such that

$$b(u,v) = (u, Bv)$$

where (f,g) denotes the Hilbert space inner product of f and g.

5. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ and $C : \mathbb{R} \to \mathbb{R}^{n\times k}$ be continuous matrix valued mappings and $u : \mathbb{R} \to \mathbb{R}^m$ a continuous vector valued function. An input-output system $\mathcal{S}$ is an initial value problem

$$\dot{x} = A(t)x + B(t)u$$

$$x(0) = 0$$

together with an ouput relation

$$v = C(t)x \ .$$

The system $\mathcal{S}$ is said to be *I-O stable* if there is a constant $K > 0$ such that

$$\sup_{t\ge 0} |v(t)| \le K \sup_{t\ge 0} |u(t)|$$

(a) Let Φ be a fundamental matrix normalized at $t_0 = 0$ for the system $\dot{x} = A(t)x$, and define $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{k\times m}$ by

$$h(t,s) = C(t)\Phi(t)\Phi^{-1}(s)B(s) \ .$$

Show that, if there exists $M > 0$ such that

$$\int_0^t |h(t,s)|\, ds \le M \quad \text{for all} \quad t \ge 0 \ .$$

then the system $\mathcal{S}$ is I-O stable .

(b) Next suppose that A, B, and C are constant matrices. Show that, if all of the eigenvalues of A have negative real part, then the system $\mathcal{S}$ is I-O stable .

(c) Finally, assume that A, B, and C are constant and also that B and C are invertible. In this case, show that, the system $\mathcal{S}$ is I-O stable if and only if all of the eigenvalues of A have negative real part.

6. Let $f(x)$ be a continuous bounded (i.e. there is a constant M such that $|f(x)| \leq M \; \forall \; x \in \mathbb{R}$) function on the real line . Prove that there is a unique bounded C^2 solution u of the differential equation

$$\left(\frac{d}{dx} - 1\right)\left(\frac{d}{dx} + 2\right) u = f$$

Writing an explicit formula for u as an integral or a sum of integrals will help to show that u is bounded.

7. Let $q(x)$ be a continuous real-valued function on the interval $[0,1]$. Consider the Sturm-Liouville problem

$$u'' + q(x)u = \lambda u$$
$$u'(0) = u(0) \qquad u(1) = 0$$

Show that all eigenvalues are real, simple (i.e. if two eigenfunctions have the same eigenvalue, then one is a constant multiple of the other), and satisfy

$$\lambda \leq \max_{x \in [0,1]} q(x)$$

Hint: Prove and use the inequality $|u(0)|^2 \leq \int_0^1 |u'(t)|^2 dt$

8. a) Show that, for any tempered distribution $\mu \in S'(\mathbb{R}^n)$, there is a unique tempered distribution $\omega \in S'(\mathbb{R}^n)$ that solves

$$\Delta\omega - \omega = \mu$$

b) If μ is the distribution

$$\langle \mu, \phi \rangle = \int_{-\infty}^{\infty} \phi(0,y)dy$$

show that ω is a function, defined at every point $(x,y) \in \mathbb{R}^2$ and bounded by a constant.

c) For each $0 \leq \theta < 2\pi$, calculate

$$\lim_{r \to \infty} \omega(r\cos(\theta), r\sin(\theta))$$