# LINEAR ANALYSIS PRELIM EXAM

## Autumn 2009

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. (a) Let $\lambda_{\max}(A)$ denote the maximal eigenvalue of the real symmetric $n \times n$ matrix $A$. Show that $\lambda_{\max}$ is a Lipschitz function of $A$.
   (b) Analogs of this result fail if the assumption that $A$ is symmetric is removed. Show that if $n > 1$, then $\lambda_{\max}$ is not a Lipschitz function on the space of real $n \times n$ matrices all of whose eigenvalues are real.

2. Let $T > 0$ and $x_0 \in \mathbb{R}^n$. Suppose that $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies

$$|f(t,x) - f(t,y)| \leq C|x-y||\log|x-y||$$

for some $C > 0$. Show uniqueness for the initial value problem

$$\begin{cases} x' = f(t,x) \\ x(0) = x_0 \end{cases}.$$

That is, if $x(t)$, $y(t)$ are $C^1$ maps of $[0,T]$ into $\mathbb{R}^n$ which solve this initial value problem, then $x(t) = y(t)$ for $0 \leq t \leq T$.

3. Let $A : [0,1] \to \mathbb{C}^{n\times n}$ be continuous and suppose that for each $t \in [0,1]$, all eigenvalues of $A(t)$ satisfy $\operatorname{Re}\lambda > 0$. Show that there is a continuous $B : [0,1] \to \mathbb{C}^{n\times n}$ such that

$$(B(t) + I)(B(t) - 2I) = A(t)$$

for all $t \in [0,1]$.

4. Consider the explicit one-step scheme

$$x_{k+1} = x_k + h\psi(h, t_k, x_k)$$

with step size $h > 0$ for numerically approximating the solution $x(t)$ of the initial value problem

$$x'(t) = f(t,x), \qquad x(0) = x_0$$

on the interval $[0,T]$. Here $t_k = kh$ for $0 \leq kh \leq T$ and $x_k$ represents the approximation to $x(t_k)$. Assume that $x, f, \psi$ are $\mathbb{R}^n$-valued, $f$ is continuous in $t, x$ and uniformly Lipschitz continuous in $x$ on $[0,T] \times \mathbb{R}^n$, and $\psi$ is continuous in $h, t, x$ and uniformly Lipschitz continuous in $x$ on $[0,h_0] \times [0,T] \times \mathbb{R}^n$ for some $h_0 > 0$.
   (a) Define what it means for the scheme to be accurate of order $p$, for a positive integer $p$.
   (b) Show that if the scheme is accurate of order at least 1, then the numerical approximation defined by the scheme converges to the actual solution in the sense that

$$\max_{0\leq kh\leq T} |x(t_k) - x_k| \to 0$$

as $h \to 0$.

5. Let $C_0([0,1])$ denote the vector space of complex-valued, continuous functions $f$ on $[0,1]$ satisfying $f(0) = f(1) = 0$. For $f \in C_0([0,1])$, the restriction $f|_{(0,1)}$ defines a distribution $\in \mathcal{D}'((0,1))$, whose distribution derivative we denote by $f' \in \mathcal{D}'((0,1))$.

   Define
$$H^1 = \{f \in C_0([0,1]) : f' \in L^2((0,1))\},$$
   where $L^2((0,1))$ is viewed as a subspace of $\mathcal{D}'((0,1))$ in the usual way. For $f$, $g \in H^1$, set
$$\langle f, g\rangle_1 = \int_0^1 f'(x)\overline{g'(x)}\, dx.$$
   (a) Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on $H^1$ with respect to which $H^1$ is complete.
   (b) Show that for each $x \in (0,1)$, the linear functional
$$\ell_x(f) = f(x)$$
   is bounded on $H^1$.
   (c) For each $x \in (0,1)$, find the unique element $g_x \in H^1$ so that
$$\ell_x(f) = \langle f, g_x\rangle_1 \quad \text{for all } f \in H^1.$$

6. Define the projection operator $P$ on $L^2(S^1)$ in terms of the Fourier representation by
$$P\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{\infty} a_n e^{in\theta}.$$
   For $g \in C(S^1, \mathbb{C})$, define the multiplication operator $M_g$ on $L^2(S^1)$ by $M_g f = gf$. Define the commutator $C_g : L^2(S^1) \to L^2(S^1)$ by
$$C_g = [P, M_g] \equiv PM_g - M_g P.$$
   (a) Identify ${C_g}^*$. Deduce that if $g$ is real-valued, then $C_g$ is skew-Hermitian.
   (b) If $g$ is a trigonometric polynomial
$$g = \sum_{n=-N}^{N} b_n e^{in\theta},$$
   show that $C_g$ is an operator of finite rank.
   (c) Show that if $g \in C(S^1, \mathbb{C})$, then $C_g$ is a compact operator on $L^2(S^1)$. (*Hint*: Approximate $g$.)

7. Solve explicitly the Cauchy problem for the heat equation:
$$\begin{cases} u_t = u_{xx} \quad \text{for } (x,t) \in \mathbb{R}^2 \\ u(0,t) = 1 + \sin t \\ u_x(0,t) = \sin\left(\frac{\pi}{4} + t\right) \end{cases}$$
   for a function $u(x,t)$, assumed to be $2\pi$-periodic in $t$.
   (*Caution:* Note that the "initial conditions" are given at $x = 0$, not $t = 0$.)

8. Consider the linear functionals $u_1$, $u_2$ defined by

$$u_1(\varphi) = \int_{-\infty}^{\infty} \varphi(x, 0)\, dx,$$

$$u_2(\varphi) = \int_{-\infty}^{\infty} \varphi(0, y)\, dy,$$

for $\varphi$ a suitable function on $\mathbb{R}^2$.

(a) Show that $u_1$, $u_2 \in \mathcal{S}'(\mathbb{R}^2)$ (the space of tempered distributions).

(b) Show that $\widehat{u_1} = 2\pi u_2$, where $\widehat{\ }$ denotes the Fourier transform.

Be sure to justify your arguments.