Linear Analysis Prelim Exam

Autumn 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation: For a vector $v \in \mathbb{R}^n$ or $\mathbb{C}^n$, $|v|$ denotes the Euclidean norm $\left(\sum_{j=1}^n |v_j|^2\right)^{1/2}$.

1. Find all solutions $u \in \mathcal{D}'(\mathbb{R})$ to the equation $\quad x\,\partial_x u = 1$.

Be careful to show that your solutions do in fact solve the equation, and that they represent all distribution solutions to the equation.

2. For this problem, $\mathcal{V} = \{v_1, v_2, \dots\}$ is assumed to be a countable collection of orthonormal vectors in a complex Hilbert space $\mathcal{H}$ with inner product $\langle\,\cdot\,,\cdot\,\rangle$ and norm $\|\cdot\|$.

The collection $\mathcal{V}$ is said to be *complete* if the only vector $w \in \mathcal{H}$ that is orthogonal to all $v_j \in \mathcal{V}$ is the 0 vector. Show that each of the following conditions on $\mathcal{V}$ is equivalent to $\mathcal{V}$ being complete.

a) For every $w \in \mathcal{H}$, the following holds

$$\|w\|^2 = \sum_{j=1}^{\infty} \left|\langle w, v_j\rangle\right|^2$$

b) If $w \in \mathcal{H}$, and $\epsilon > 0$, there exists $v \in \operatorname{span}(\mathcal{V})$ such that $\|v - w\| < \epsilon$.

3. Let A be the matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -2 \end{bmatrix}$$

a) Find the Jordan decomposition of A.

b) Find a fundamental matrix for the initial value problem $x' = Ax$, $x(0) = x_0$.

c) Let $f : \mathbb{R} \to \mathbb{R}^3$ be a continuous vector-valued function with $\int_0^\infty |f(t)|\,dt < \infty$. Let $x(t)$ be the solution to the following initial value problem for $t \geq 0$,

$$x'(t) = Ax(t) + f(t)\,, \qquad x(0) = 0\,.$$

Show there exists a constant vector $v \in \mathbb{R}^3$ such that $|x(t) - v| \to 0$ as $t \to +\infty$, and calculate v in terms of f. (Your answer is allowed to involve the inverse of a matrix.)

4. Given sequences of real numbers $a_1, a_2, a_3 \ldots$ and $b_1, b_2, b_3, \ldots$, for each $n \geq 1$ define the $n \times n$ real tridiagonal symmetric matrix T_n as follows:

$$T_n = \begin{bmatrix} a_n & b_{n-1} & & \\ b_{n-1} & a_{n-1} & \ddots & \\ & \ddots & \ddots & b_1 \\ & & b_1 & a_1 \end{bmatrix}$$

a) Let $p_n(x)$ be the characteristic polynomial of T_n. Find an expression for $p_n(x)$ in terms of a_n, b_{n-1}, $p_{n-1}(x)$ and $p_{n-2}(x)$.

b) The only eigenvalue of T_1 is a_1. Show that if $b_1 \neq 0$, then the two eigenvalues $\lambda_1 \geq \lambda_2$ of T_2 have the property that $\lambda_1 > a_1 > \lambda_2$.

c) Assume that $b_i \neq 0$ for all $1 \leq i \leq n-1$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of T_n, and let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1}$ be the eigenvalues of T_{n-1}. Use induction and part (a) to show that the μ's and λ's *interlace*, i.e. $\lambda_1 > \mu_1 > \lambda_2 > \ldots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$.

5. Show that there is a unique (up to equality almost everywhere) function $g \in L^2([-\pi, \pi])$ which minimizes the quantity

$$\int_{-\pi}^{\pi} \left| e^t - g(t) \right|^2 dt$$

subject to the conditions

$$\begin{cases} \widehat{g}(0) = 1\,, \\ \widehat{g}(-k) = -\widehat{g}(k) \text{ for all integers } k \neq 0\,, \end{cases}$$

and explicitly identify the minimizing function $g(t)$. Here,

$$\widehat{g}(k) = \int_{-\pi}^{\pi} e^{-ikt}\, g(t)\, dt\,.$$

6. a) Given an $m \times n$ complex matrix A, where $m \geq n$, and an integer $k \leq n$, calculate the value

$$\inf_{\operatorname{rank}(B) \leq k} ||A - B||\,,$$

where the norm $||\cdot||$ denotes the spectral norm,

$$||A|| = \sup_{|x|=1} |Ax|\,,$$

and the infimum is taken over complex $m \times n$ matrices B of rank at most k. The answer should come as a function of well-known values associated with A.

b) What conditions must one impose on the matrix A so that the minimizer matrix B you found in part (a) (for a fixed k) is unique?

7. Let $f(x)$ be a C^∞ function on the real line, which is periodic with period 2π. Use Fourier series to express the solution to the initial value problem

$$\begin{cases} u_t(x,t) = iu_{xx}(x,t)\,, & x \in [0,2\pi]\,,\ \ t \geq 0\,, \\ u(x,0) = f(x)\,, & x \in [0,2\pi]\,, \end{cases}$$

with periodic boundary conditions

$$\begin{cases} u(0,t) = u(2\pi,t) \\ u_x(0,t) = u_x(2\pi,t) \end{cases} \qquad t \geq 0\,.$$

a) Show that, for all $t \geq 0$,

$$\int_0^{2\pi} |u(x,t)|^2\,dx = \int_0^{2\pi} |f(x)|^2\,dx\,.$$

b) Show that

$$\lim_{t\to 0+} \int_0^{2\pi} \big|u(x,t) - f(x)\big|^2\,dx = 0\,.$$

8. Given the initial value problem $y'(t) = f(y(t))$, $0 \leq t \leq T$, $y(0) = y_0$, where f is C^∞ in y, we investigate numerical methods of the form

$$y_{n+1} = y_n + h \cdot (Ak_1 + Bk_2)\ , \tag{1}$$

where $k_1 = f(y_n)$ and $k_2 = f(y_n + hk_1)$, for $n = 0$ through $N-1$, and $h = T/N$.

a) Find the constants A and B such that the method is accurate of as high an order as possible.

b) Consider the case of a stiff equation, i.e. $y' = ky$, $y(0) = y_0$, with $k < 0$ a negative real constant, over the interval $[0,\infty)$. We define the sequence y_n recursively by (1), with h being the stepsize parameter.
For general $h > 0$, determine an explicit expression for y_n using the values of A and B found in part (a). Find explicitly (in terms of k) the set of values of h for which the sequence $\{y_n\}$ stays bounded as $n \to \infty$.