• Do as many of the eight problems as you can.
• Four completely correct solutions will be a pass;
• A few complete solutions will count more than many partial solutions. Always carefully justify your answers.
• If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation

• $\mathbb{R}^{n \times m}$ denotes the $n \times m$ real matrices.
• $\mathbb{C}^{n \times m}$ denotes the $n \times m$ complex matrices.
• $\mathcal{S}^n$ denotes the real symmetric matrices.
• $H \in \mathcal{S}^n$ is said to be positive definite ($H > 0$) if $x^T H x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.
• $H \in \mathcal{S}^n$ is said to be positive semi-definite ($H \geq 0$) if $x^T H x \geq 0$ for all $x \in \mathbb{R}^n$.
• The singular values of a matrix $A$ are the eigenvalues of the matrix $\sqrt{A^* A}$.
• $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of smooth rapidly decaying functions.
• $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.
• A function $f \in L^1(\mathbb{R}^n)$ acts as a tempered distribution on $\phi \in \mathcal{S}$ by
  \[ \langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x) \phi(x) d^n x \]
• $H^2(\mathbb{R}^2)$ is the closure of $C_0^\infty(\mathbb{R}^2)$ in the norm,
  \[ ||u||_{H^2(\mathbb{R}^2)}^2 := ||u||_{L^2}^2 + ||\Delta u||_{L^2}^2 \]
• In $\mathbb{R}^2$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
1. Let \( A \in \mathbb{R}^{m \times n} \), \( W \in \mathbb{R}^{m \times m} \), and \( V \in \mathbb{R}^{n \times n} \) with \( W \) and \( V \) symmetric.
(a) Show that \( V \) is positive definite on \( \ker A \), i.e.,
\[
    u^T V u > 0 \quad \text{whenever} \quad u \neq 0 \text{ and } u \in \ker A,
\]
if and only if there is a \( \kappa > 0 \) such that the matrix \( V + \kappa A^T A \) is positive definite.
(b) Suppose \( V \) is positive semidefinite on \( \ker A \), i.e.,
\[
    u^T V u \geq 0 \quad \text{whenever} \quad u \in \ker A.
\]
Show that the matrix \( M := \begin{bmatrix} V & A^T \\ A & 0 \end{bmatrix} \) is nonsingular if and only if \( V \) is positive definite on \( \ker A \) and the rank of \( A \) is \( m \).
(c) Show that the matrix
\[
    T := \begin{bmatrix} V & A^T \\ A & W \end{bmatrix}
\]
is positive definite if and only if the matrices \( V \) and \( W - AV^{-1} A^T \) are positive definite.

2. Let \( A \in \mathbb{C}^{n \times n} \) and \( \epsilon > 0 \). Show that the three sets \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) defined below are equal.
\[
\mathcal{A} := \{ \lambda \in \mathbb{C} \mid \lambda \in \Lambda(X), \| A - X \| \leq \epsilon \},
\]
\[
\mathcal{B} := \{ \lambda \in \mathbb{C} \mid \| (A - \lambda I)^{-1} \| \geq \epsilon^{-1} \text{ or } (A - \lambda I) \text{ is singular} \},
\]
\[
\mathcal{C} := \{ \lambda \in \mathbb{C} \mid \sigma_{\min}(A - \lambda I) \leq \epsilon \},
\]
where the we have used the operator 2-norm and \( \sigma_{\min}(A - \lambda I) \) is the smallest singular value of \( (A - \lambda I) \).

3. Let \( m(x) \in C^1([0, 1]) \) and \( \lambda \in \mathbb{C} \). Consider the boundary value problem
\[
    \left( \frac{d}{dx} + m(x) - \lambda \right) u = f
\]
\[
u(0) = u(1)
\]
Let \( G(\lambda) \) denote the solution operator as a mapping from \( f \in L^2(0, 1) \) to \( u \in L^2(0, 1) \).
(a) Find an explicit formula for \( G(\lambda)f \).
(b) Find an explicit formula for the eigenvalues of the boundary value problem (i.e. the values of \( \lambda \) for which \( G(\lambda) \) does not exist).
(c) Prove that, if \( \lambda \) is not an eigenvalue, \( G(\lambda) \) is a compact operator which maps \( L^2(0, 1) \) to itself.
4. Suppose $U : \mathbb{R} \rightarrow \mathbb{R}$ is $C^1$. If we interpret $U$ as the potential energy of a particle at position $x$, then $-U'(x)$ (minus the derivative of $U$) is the force acting on the particle, so (for a particle with mass 1) Newton’s law of motion is the second order ODE
\[ \frac{d^2 x}{dt^2} = -U'(x(t)) . \] (1)

For example, for a simple harmonic oscillator (spring), we could have $U(x) = \frac{1}{2}kx^2$ for some $k > 0$.

(a) Rewrite equation (1) as a first-order system.
(b) The kinetic energy of a particle is $\frac{1}{2}(\dot{x})^2$, so the total energy is
\[ E(t) = \frac{1}{2}(\dot{x}(t))^2 + U(x(t)) . \]
where the dot $\dot{} = \frac{d}{dt}$ means derivative with respect to time. Show that if $x$ solves (1), then $E(t)$ is constant, i.e., energy is conserved.
(c) Suppose that $U$ is bounded from below (that is, there exists $C \in \mathbb{R}$ such that $U(x) \geq C$ for all $x \in \mathbb{R}$). Prove that every solution of (1) exists for all time ($t \rightarrow \pm \infty$).
(d) Show that if $U(x) = -x^4$, then the solution of (1) satisfying the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$ blows up in finite time.

5. Consider the map
\[ Mf = \int_0^1 |x - y|f(y)dy \]
mapping $L^2(0,1)$ into continuous (but not necessarily bounded) functions on the real line.

(a) Show that the image of the unit ball in $L^2(0,1)$ are uniformly Lipschitz continuous; i.e.
\[ |Mf(a) - Mf(b)| \leq C|a - b| \]
where $C$ depends only on $||f||_{L^2(0,1)}$.
(b) Find the codimension 2 subspace of $L^2(0,1)$ that maps into $L^2(\mathbb{R})$.

Hint: On this subspace, $Mf$ is identically zero outside $(0,1)$.
(c) Show that $M$ is a compact and injective operator from this subspace into $L^2(\mathbb{R})$. 

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6. Prove the existence of a solution $u \in H^2(\mathbb{R}^2)$ that satisfies
$$\Delta u - u = F(x, u)$$
under the hypotheses that the $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^1)$ satisfies
$$||F(x, 0)||_{L^2} \text{ is sufficiently small}$$
$$|F(x, a) - F(x, b)| \leq (|a| + |b|) |a - b| \quad \text{for all real } x, a, b$$

*Hint: Use Fourier transform to estimate the $H^2$ norm of the solution $u$ to the linear PDE $\Delta u - u = f(x)$ in in terms of the $L^2$ norm of $f$; then define a mapping from a ball in $H^2$ to itself, and show its a contraction if $||F(x, 0)||_{L^2}$ is small enough. The fact that the supremum ($L^\infty(\mathbb{R}^2)$) norm is bounded by a constant times the $H^2(\mathbb{R}^2)$ norm may be useful. This fact is a form of the Sobolev embedding theorem.*

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7. The two definitions below describe how the tempered distributions, $r_P$ and $r_D$, act on $\phi \in \mathcal{S}$. Prove that the two definitions define the same tempered distribution.

$$\langle r_P, \phi \rangle := \lim_{\epsilon \downarrow 0} \int_{|x| > \epsilon} \frac{1}{x} \phi(x) dx$$

$$\langle r_D, \phi \rangle := \langle \partial_x \log(|x|), \phi \rangle$$

Part of the problem is to prove that each does indeed define a tempered distribution.

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8. (a) Prove that the linear map
$$T\phi = \phi(|x|^2) \quad x \in \mathbb{R}^2$$
maps $\mathcal{S}(\mathbb{R}^1)$ to $\mathcal{S}(\mathbb{R}^2)$.

(b) By duality, $T$ induces a map
$$T^*: \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^1)$$
Every $f \in \mathcal{S}(\mathbb{R}^2)$ defines a distribution in $\mathcal{S}'(\mathbb{R}^2)$. For such a distribution, $T^*f$ is also a function in $\mathcal{S}'(\mathbb{R}^1)$. Find an explicit expression for the function $T^*f$ (i.e. write a formula for its value at every $t \in \mathbb{R}$, not just a formula for its action as a distribution). *Hint: Use polar coordinates*