LINEAR ANALYSIS PRELIM EXAM

Autumn 2015

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Let $f : \mathbb{N} \to \mathbb{C}$ be a bounded function, where $\mathbb{N}$ denotes the set of positive integers. Suppose $1 \le p < \infty$ and define $M_f : \ell^p \to \ell^p$ by

$$M_f(\{a_n\}_{n=1}^\infty) = \{f(n)a_n\}_{n=1}^\infty.$$

   (a) Find (with proof) the spectrum of M_f.
   (b) State and prove a necessary and sufficient condition in terms of f for M_f to be a compact operator.
   (c) Is it the case that for all f the spectrum of M_f consists entirely of eigenvalues? Prove that this holds or provide a counterexample.

2. Let $M \in \mathbb{C}^{n\times n}$ be an $n \times n$ matrix. Given $b \in \mathbb{C}^n$ and $x_0 \in \mathbb{C}^n$, generate the sequence $\{x_k\}_{k=0}^\infty$ by the iteration

$$x_{k+1} = b + Mx_k.$$

Give (and prove) necessary and sufficient conditions on the matrix M such that, for any given $b \in \mathbb{C}^n$ and $x_0 \in \mathbb{C}^n$, the sequence x_k converges.

3. Let $A \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix of rank r with $m > n > r > 0$, and fix a vector $b \in \mathbb{R}^m$. For each $\lambda > 0$, define

$$x_\lambda = (A^TA + \lambda I)^{-1}A^Tb.$$

   (a) Use the singular value decomposition (SVD) of A to write x_λ as a linear combination of the right singular vectors of A, with coefficients which are explicit expressions in terms of b, the left singular vectors of A, and the singular values of A.
   (b) Show that the limit $\lim_{\lambda\to 0^+} x_\lambda$ exists.
   (c) Define $x_0 \equiv \lim_{\lambda\to 0^+} x_\lambda$ to be this limit. Show that x_0 is a solution of the least squares problem

$$\min_{x\in\mathbb{R}^n} |b - Ax|^2,$$

   where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$ (i.e., the ℓ^2 norm on $\mathbb{R}^m$).

4. Suppose H is a Hilbert space. Recall that $U \in \mathcal{L}(H)$ is unitary if $U^*U = UU^* = I$.
   (a) Show that if $U \in \mathcal{L}(H)$ is unitary, then $H = \overline{\mathrm{Ran}(I - U)} \oplus \mathrm{Ker}(I - U)$.
   (b) Let P be the orthogonal projection onto $\mathrm{Ker}(I - U)$. Set $S_n = \frac{1}{n}\sum_{j=0}^{n-1} U^j$. Show that if $x \in H$, then $\|S_n x - Px\| \to 0$ as $n \to \infty$.

5. Let $A \in \mathbb{R}^{n\times n}$ be skew-symmetric. Let $M : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and satisfy

$$|M(t,x)| \le \frac{C}{1+t^2}|x|$$

for some $C > 0$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$. Show that the initial value problem

$$x' = Ax + M(t,x), \qquad x(0) = x_0$$

has a solution $x(t)$ defined for $t \in [0,\infty)$, and show that $x(t)$ is bounded.

6. Recall that the convolution of two functions p and q defined on $\mathbb{R}$ is given by

$$(p * q)(x) = \int_{\mathbb{R}} p(x-y)q(y)\,dy \qquad \text{for } x \in \mathbb{R}.$$

Suppose $\varphi \in \mathcal{S}(\mathbb{R})$ satisfies $\varphi \geq 0$, $\int_{\mathbb{R}} \varphi(x)\,dx = 1$, and $\int_{\mathbb{R}} x\varphi(x)\,dx = 0$. Define φ_k inductively by $\varphi_1 = \varphi$, $\varphi_k = \varphi_{k-1} * \varphi$, $k \geq 2$. Show that $\lim_{k\to\infty} k\varphi_k(kx) = \delta_0$, where the limit is in $\mathcal{S}'(\mathbb{R})$, and δ_0 is the distribution $\langle \delta_0, \psi \rangle = \psi(0)$ for $\psi \in \mathcal{S}(\mathbb{R})$.

7. Let $A \in \mathbb{C}^{n\times n}$ be an $n \times n$ matrix for which the geometric multiplicity of the eigenvalue $\lambda = 0$ equals its algebraic multiplicity. Show that there exists a matrix $B \in \mathbb{C}^{n\times n}$ for which $B^2 = A$.

8. Solve the following Dirichlet problem:

$$\begin{array}{lll} u_{xx} + y^2 u_{yy} + y u_y = 0 & \text{for} & (x,y) \in (0,\pi) \times (0,1), \\ u(0,y) = u(\pi,y) = 0 & \text{for} & y \in [0,1], \\ u(x,0) = 0 & \text{for} & x \in [0,\pi], \\ u(x,1) = 5\sin 2x - \sin 3x & \text{for} & x \in [0,\pi]. \end{array}$$