Do as many of the eight problems as you can. Four problems done correctly will be a clear
pass. Always carefully justify your answers. In partial solutions, it helps to indicate where
the gaps are. All manifolds are assumed to be smooth \((C^\infty)\) and without boundary, and all
structures on them (e.g. differential forms, vector fields) are assumed to be smooth.

1. Suppose \(X, Y,\) and \(Z\) are path-connected spaces, \(p: Y \to Z\) is a covering map, and
\(f: X \to Z\) is continuous. Let
\[
W = \{ (x, y) \in X \times Y : f(x) = p(y) \}.
\]
Prove:
   a) If \(f_*: \pi_1(X) \to \pi_1(Z)\) is surjective, then \(W\) is path-connected.
   b) If \(Y\) is simply-connected, then the converse of part (a) is also true.

2. Let \(X\) be the bouquet of two copies of the circle \(S^1,\) and let \(n > 2.\) Give an explicit
description of a covering map \(p: Y \to X\) such that the image of \(p_*: \pi_1(Y, \ast) \to \pi_1(X, \ast)\) is
a free group on \(n\) generators and is a normal subgroup of index \(n - 1.\) (You may describe \(Y\)
by drawing a picture, but you must justify why \(p: Y \to X\) has the requisite properties.)

3. Using "bump" functions, prove that any compact \(n\)-manifold may be smoothly em-
bedded in \(\mathbb{R}^N\) for \(N\) sufficiently large.

4. Let \(H\) be a smooth monoid; that is, \(H\) is a manifold and a monoid such that the
multiplication map \(H \times H \to H\) is smooth. (Recall that a monoid is a set with an associative
multiplication and a unit.) Prove that the set of invertible elements in \(H\) is open and that
the map \(g \to g^{-1}\) is smooth on this open subset.
   Hint: Observe that it suffices to show that the identity has a neighborhood consisting of
invertible elements on which the inverse map is smooth.

5. Prove that every bounded vector field on \(\mathbb{R}^n\) is complete.

6. Let \(w = f dx + g dy\) be a closed one-form on \(R^2 - \{0\}\) and suppose that \(f\) and \(g\) are
bounded. Prove that \(w\) is exact.

7. Let \(\omega\) be a nowhere vanishing one-form on the \(n\)-manifold \(M, n > 1.\) Let \(D\) be the
distribution defined by \(D_p = \ker \omega_p, p \in M.\) Prove that \(D\) is integrable if and only if, for
every \(p \in M,\) there is a neighborhood \(U\) of \(p\) and a smooth nowhere zero real-valued function
\(f\) on \(U\) such that \(f \omega\) is exact on \(U.\)

8. Show that the Lie algebra of the Lie group \(GL(n, \mathbb{R})\) is isomorphic to the vector space
\(M_n(\mathbb{R})\) of all \(n \times n\) real matrices with Lie bracket given by
\[
[A, B] = AB - BA,
\]
for \(A, B \in M_n(\mathbb{R}).\)