Topology and Geometry of Manifolds Preliminary Exam
September 14, 2006

Do as many of the eight problems as you can. Four problems done correctly will be a clear pass. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word "smooth" means C^∞, and all manifolds are assumed to be smooth and without boundary unless otherwise specified.

1. Let $T^n = S^1 \times \cdots \times S^1$ denote the n-torus, and let M be a connected topological manifold with finite fundamental group. Show that any continuous map from M to T^n is homotopic to a constant map.

2. Represent the Möbius strip as the quotient space of the square $[-1, 1] \times [-1, 1]$ by identifying $(1, y)$ with $(-1, -y)$ for $-1 \leq y \leq 1$. Its boundary (as a manifold with boundary) is the image of the set $\{(x, y) \in [-1, 1] \times [-1, 1] : |y| = 1\}$ under the quotient map. Prove that there does not exist a retraction from the Möbius strip to its boundary.

3. Let X be a complete, smooth vector field on $\mathbb{R}^2$, and let ϕ denote its flow. We say X is *area-preserving* if $\phi_t^*(dA) = dA$ for all t, where $dA = dx \wedge dy$ is the standard area form. Show that X is area-preserving if and only if there exists a function $f \in C^\infty(\mathbb{R}^2)$ such that
$$X = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}.$$
[Hint: Think about the Lie derivative of dA with respect to X.]

4. For each of the following vector fields on $\mathbb{R}$, find the flow and determine whether the vector field is complete.
$$X = x\frac{\partial}{\partial x},$$
$$Y = x^2\frac{\partial}{\partial x}.$$

5. Define a subset $S \subset \mathbb{R}^4$ by
$$S = \{(x, y, u, v) : x^2 + y^2 - 2uv = x^2 - y^2 + u^2 - v^2 = 0\}.$$
   (a) Prove that there exist open sets $U, V \subset \mathbb{R}^2$ such that $(-1, 1, 1, 1) \in U \times V$, and smooth functions $\alpha, \beta \colon U \to \mathbb{R}$ with the following property: $(x, y, u, v) \in S \cap (U \times V)$ if and only if $(x, y) \in U$, $u = \alpha(x, y)$, and $v = \beta(x, y)$.

   (b) Compute the following partial derivatives at $(x, y) = (-1, 1)$:
$$\frac{\partial\alpha}{\partial x}, \quad \frac{\partial\alpha}{\partial y}, \quad \frac{\partial\beta}{\partial x}, \quad \frac{\partial\beta}{\partial y}.$$

6. Suppose (M, g) is a Riemannian manifold, and $f\colon M \to \mathbb{R}$ is a smooth proper map such that $|\operatorname{grad} f|_g \equiv 1$. (Recall that a map is *proper* if the inverse image of every compact set is compact.)

   (a) If ϕ is the flow of $\operatorname{grad} f$, show that $f(\phi_t(x)) = t + f(x)$ whenever $(t, x) \in \mathbb{R} \times M$ is in the domain of ϕ.

   (b) Show that $\operatorname{grad} f$ is complete.

7. The *Heisenberg group* is the Lie group whose underlying manifold is $\mathbb{R}^3$, with the following group structure:

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy' - yx').$$

   (You may accept without proof that it is a Lie group. Its identity element is $(0, 0, 0)$, and the inverse of (x, y, z) is $(-x, -y, -z)$.)

   (a) Compute the left-invariant vector fields X, Y, Z whose values at the identity are

$$\begin{aligned} X|_{(0,0,0)} &= \frac{\partial}{\partial x}, \\ Y|_{(0,0,0)} &= \frac{\partial}{\partial y}, \\ Z|_{(0,0,0)} &= \frac{\partial}{\partial z}. \end{aligned}$$

   (b) Show that the distribution spanned by X and Z is integrable, but the one spanned by X and Y is not.

8. Let M be a smooth, oriented, compact n-manifold without boundary and let I be the interval $[0, 1]$. Suppose α is a p-form and β is an $(n-p)$-form, both defined and smooth on $M \times I$. Prove the following "integration-by-parts formula":

$$\int_{\boxed{M\times I}} d\alpha \wedge \beta = \left[\int_M \alpha \wedge \beta \right]_{t=0}^{t=1} - (-1)^p \int_{\boxed{M\times I}} \alpha \wedge d\beta.$$

   (Here t is the coordinate on I.)

should be I x M