Do as many of the eight problems as you can. Four problems done correctly will be a clear pass. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word “smooth” means $C^\infty$, and all manifolds are assumed to be smooth and without boundary unless otherwise specified.

1. Let $T^n = S^1 \times \cdots \times S^1$ denote the $n$-torus, and let $M$ be a connected topological manifold with finite fundamental group. Show that any continuous map from $M$ to $T^n$ is homotopic to a constant map.

2. Represent the Möbius strip as the quotient space of the square $[-1, 1] \times [-1, 1]$ by identifying $(1, y)$ with $(-1, -y)$ for $-1 \leq y \leq 1$. Its boundary (as a manifold with boundary) is the image of the set $\{(x, y) \in [-1, 1] \times [-1, 1] : |y| = 1\}$ under the quotient map. Prove that there does not exist a retraction from the Möbius strip to its boundary.

3. Let $X$ be a complete, smooth vector field on $\mathbb{R}^2$, and let $\phi$ denote its flow. We say $X$ is area-preserving if $\phi^t_\ast (dA) = dA$ for all $t$, where $dA = dx \wedge dy$ is the standard area form. Show that $X$ is area-preserving if and only if there exists a function $f \in C^\infty(\mathbb{R}^2)$ such that

$$X = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial y}.$$  

[Hint: Think about the Lie derivative of $dA$ with respect to $X$.]

4. For each of the following vector fields on $\mathbb{R}$, find the flow and determine whether the vector field is complete.

$$X = x \frac{\partial}{\partial x},$$

$$Y = x^2 \frac{\partial}{\partial x}.$$

5. Define a subset $S \subset \mathbb{R}^4$ by

$$S = \{(x, y, u, v) : x^2 + y^2 - 2uv = x^2 - y^2 + u^2 - v^2 = 0\}.$$

(a) Prove that there exist open sets $U, V \subset \mathbb{R}^2$ such that $(-1, 1, 1, 1) \in U \times V$, and smooth functions $\alpha, \beta : U \to \mathbb{R}$ with the following property: $(x, y, u, v) \in S \cap (U \times V)$ if and only if $(x, y) \in U$, $u = \alpha(x, y)$, and $v = \beta(x, y)$.

(b) Compute the following partial derivatives at $(x, y) = (-1, 1)$:

$$\frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial y}, \frac{\partial \beta}{\partial x}, \frac{\partial \beta}{\partial y}.$$
6. Suppose $(M, g)$ is a Riemannian manifold, and $f: M \rightarrow \mathbb{R}$ is a smooth proper map such that $|\text{grad } f|_g \equiv 1$. (Recall that a map is proper if the inverse image of every compact set is compact.)

(a) If $\phi$ is the flow of $\text{grad } f$, show that $f(\phi_t(x)) = t + f(x)$ whenever $(t, x) \in \mathbb{R} \times M$ is in the domain of $\phi$.

(b) Show that $\text{grad } f$ is complete.

7. The Heisenberg group is the Lie group whose underlying manifold is $\mathbb{R}^3$, with the following group structure:

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy' - yx').$$

(You may accept without proof that it is a Lie group. Its identity element is $(0, 0, 0)$, and the inverse of $(x, y, z)$ is $(-x, -y, -z)$.)

(a) Compute the left-invariant vector fields $X, Y, Z$ whose values at the identity are

$$X|_{(0,0,0)} = \frac{\partial}{\partial x},$$

$$Y|_{(0,0,0)} = \frac{\partial}{\partial y},$$

$$Z|_{(0,0,0)} = \frac{\partial}{\partial z}.$$

(b) Show that the distribution spanned by $X$ and $Z$ is integrable, but the one spanned by $X$ and $Y$ is not.

8. Let $M$ be a smooth, oriented, compact $n$-manifold without boundary and let $I$ be the interval $[0, 1]$. Suppose $\alpha$ is a $p$-form and $\beta$ is an $(n-p)$-form, both defined and smooth on $M \times I$. Prove the following “integration-by-parts formula”:

$$\int_{M \times I} d\alpha \wedge \beta = \left[ \int_M \alpha \wedge \beta \right]_{t=0}^{t=1} - (-1)^p \int_{M \times I} \alpha \wedge d\beta.$$

(Here $t$ is the coordinate on $I$.)