Topology and Geometry of Manifolds Preliminary Exam

September 13, 2007

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word smooth means C^∞. All manifolds are assumed to be smooth and without boundary unless otherwise specified, as are all quantities related to manifolds. (E.g. all maps, vector fields, differential forms, etc. are assumed to be smooth).

(1) Consider the map

$$f : \mathbb{C}^2 \to \mathbb{R}^2 \, : \, (z, w) \mapsto \big(|z|^2 + |w|^2, |z|^2 - |w|^2\big)$$

where $\mathbb{C}$ denotes the complex numbers. Prove that $f^{-1}(1, 0) \subset \mathbb{C}^2$ is a smooth, compact, 2-dimensional submanifold.

(2) Let X be the vector field on $\mathbb{R}^n$ given by

$$X = \sum_{k=1}^{n} x^k \frac{\partial}{\partial x^k},$$

where $x = (x^1, x^2, \ldots, x^n)$ are the standard coordinates on $\mathbb{R}^n$. Let ω be the $n-1$ form on $\mathbb{R}^n_0 =: \mathbb{R}^n \setminus \{0\}$ defined by the formula

$$\omega = \frac{1}{c_n \|x\|^n} X \lrcorner \; (dx^1 \wedge \cdots \wedge dx^n),$$

where $\lrcorner$ denotes interior multiplication[1], c_n is the volume of the unit sphere in $\mathbb{R}^n$ and $\|x\|$ is the Euclidean norm on $\mathbb{R}^n$. The *index* of a smooth map $f : S^{n-1} \to \mathbb{R}^n_0$ is the real number[2]

$$\mathrm{Ind}(f) = \int_{S^{n-1}} f^* \omega \, .$$

Show that if $g : S^{n-1} \to \mathbb{R}^n_0$ is smoothly homotopic to f then $\mathrm{Ind}(f) = \mathrm{Ind}(g)$.
(Recall that g is smoothly homotopic to f if there is a smooth map

$$H : S^{n-1} \times [0, 1] \to \mathbb{R}^n_0$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in S^{n-1}$.)

(3) Suppose that X is a space whose fundamental group is the free group on three generators a, b, c. Let $Y = S^1 \vee S^1$, a bouquet of two circles, and suppose that $f : Y \to X$ maps the first circle to ab and the second circle to b^2c. Let CY be the cone on Y (that is, $CY = Y \times I / Y \times 1$) and let $Z = X \coprod CY / \sim$, where $\sim$ is the equivalence relation generated by $(y, 0) \sim f(y)$ for all $y \in Y$. Determine the fundamental group of Z and explicitly identify generators and relations.

(4) Let M and P be compact, connected manifolds, and let $f : P \to M$ be a submersion. Recall that the *fiber* of f over $x \in M$ is the smooth submanifold $P_x = f^{-1}(x)$.

(a) Let X be a smooth vector field on M. A smooth vector field $\widetilde{X}$ on P is called a *lift* of X if $f_*(\widetilde{X}_p) = X_{f(p)}$ for all $p \in P$. Prove that every vector field on M has a lift.

[1] Recall that if η is a p form then $X \lrcorner \; \eta$ is the $p-1$ form defined by $X \lrcorner \; \eta \, (Y_1, \ldots, Y_{p-1}) = \eta(X, Y_1, \ldots, Y_{p-1})$.

[2] In fact, it is an integer, but you need not show that.

(b) Let $\widetilde{X}$ be a lift of X, and let ν_t and $\widetilde{\nu}_t$, $t \in \mathbb{R}$, denote the flows of X and $\widetilde{X}$, respectively. Prove that the restriction

$$\widetilde{\nu}_t|_{P_x} : P_x \to P_{\nu_t(x)}$$

is a diffeomorphism for all $x \in M$.

(c) Using parts (a) and (b) conclude that all fibers of f are diffeomorphic.

(5) Let $h : S^1 \to S^1$ be continuous and antipode preserving (that is, $h(-x) = -h(x)$ for all $x \in S^1$). The point of this problem is to prove that h is not homotopic to a constant map.

(a) Let $p : S^1 \to S^1$ be the covering map $p(z) = z^2$, where we regard $S^1 \subset \mathbb{C}$. Prove that there exists a continuous map $g : S^1 \to S^1$ such that $p \circ h = g \circ p$.

(b) Prove that g induces an injective homomorphism of fundamental groups.

(c) Conclude that h is not homotopic to a constant map.

(6) Let M denote the set of transverse pairs (V, W) of two-dimensional linear subspaces of the vector space $\mathbb{R}^4$. (Recall that V and W are said to be *transverse* if $V + W = \mathbb{R}^4$.) Use the appropriate theorems about group actions and homogeneous spaces to prove that M has a natural topology and smooth structure making it into a smooth manifold. What is the dimension of M?

(7) Let G be a Lie group of dimension n, with identity element e. Recall that multiplication from the left by an element $h \in G$ defines a diffeomorphism

$$L_h : G \to G \;:\; g \mapsto h \cdot g\,.$$

A differential form ω is said to be *left invariant* if it satisfies the condition $L_h^*\omega = \omega$ for all $h \in G$.

(a) Prove that any covector in T^*G_e uniquely extends to a smooth, left-invariant 1-form on G.

(b) Use the result of (a) to prove that there exist n pointwise independent, left-invariant 1-forms ω^k, $k = 1, 2, \ldots, n$.

(c) Let ω^k, $k = 1, 2, \ldots, n$ be as in part (b). Prove that there exist constants $c^k_{i,j}$, such that

$$d\omega^k = \sum_{i<j} c^k_{i,j}\omega^i \wedge \omega^j$$

for all k.

(8) Let ω^k, $k = 1, 2, \ldots, n$ be pointwise independent 1-forms on $\mathbb{R}^n$ such that

$$d\omega^k = \sum_{i<j} c^k_{i,j}\omega^i \wedge \omega^j\,,$$

where $c^k_{i,j}$ are constants.

(a) Let

$$\pi_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \;:\; (x_1, x_2) \mapsto x_j \text{ for } j = 1, 2$$

and consider the distribution

$$D = \{X \in T(\mathbb{R}^n \times \mathbb{R}^n) \;:\; X \lrcorner\, \theta^k = 0,\ k = 1, 2, \ldots, n\},$$

where $\theta^k = \pi_1^*\omega^k - \pi_2^*\omega^k$. Prove that D is involutive.

(b) Choose a point $x_0 \in \mathbb{R}^n$. Using the result of (a), prove that there is a unique, maximal n-dimensional integral manifold of D containing the point $(0, x_0) \in \mathbb{R}^n \times \mathbb{R}^n$.

(c) Using the result of (b), show that there is a unique map $f : U \to \mathbb{R}^n$, defined on a neighborhood U of $0 \in \mathbb{R}^n$, such that

$$f(0) = x_0 \text{ and } f^*\omega^k = \omega^k \text{ for all } k.$$

Hint: *Look at the graph of f: $\{(x, f(x)) \;:\; x \in U\}$.*