Topology and Geometry of Manifolds Preliminary Exam
September 2008

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word "smooth" means C^∞, and all manifolds are assumed to be without boundary.

1. Which 2-manifolds M admit a covering map $\pi\colon S^2 \to M$?

2. Let f be a smooth real-valued function defined on an open subset $U \subseteq \mathbb{R}^n$. We say f is *harmonic* if
$$\sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2} = 0.$$
Show that f is harmonic if and only if for every $p \in U$ and every positive number r less than the distance from p to ∂U,
$$\sum_{i=1}^{n} (-1)^i \int_{S_r(p)} \frac{\partial f}{\partial x^i}\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = 0,$$
where $S_r(p)$ is the sphere of radius r around p, and $\widehat{dx^i}$ indicates that dx^i is omitted from the wedge product.

3. Let V be the following vector field on $M = \{(x,y,z) \in \mathbb{R}^3 : x > 0\}$:
$$V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}.$$
   (a) Determine the flow of V.
   (b) A function $f\colon M \to \mathbb{R}$ is said to be *V-invariant* if it is invariant under the flow of V, or equivalently if it is constant along the integral curves of V. Find the V-invariant functions.

4. Which (if any) of the following spaces are simply connected?
   (a) The space N^n of $n \times n$ nilpotent matrices over $\mathbb{R}$, $n \geq 1$, with the subspace topology inherited from $\mathbb{R}^{n^2}$. (A square matrix A is *nilpotent* if $A^k = 0$ for some positive integer k.)
   (b) $\mathbb{C}^n \setminus H$, where H is any complex linear subspace of dimension $n-1$.
   (c) The space $V_2\mathbb{R}^n$ of orthonormal ordered pairs of vectors in $\mathbb{R}^n$, $n \geq 4$, with the subspace topology inherited from $\mathbb{R}^n \times \mathbb{R}^n$. (Suggestion: Note that $V_2\mathbb{R}^n$ can be identified with the space of unit tangent vectors of S^{n-1}.)

5. (a) If ω is a nonvanishing smooth 1-form on a smooth manifold, show that the distribution annihilated by ω is integrable if and only if $\omega \wedge d\omega = 0$.

   (b) If X is a nonvanishing smooth vector field on $\mathbb{R}^3$, prove that the following conditions are equivalent.

      i. Every point in $\mathbb{R}^3$ has a neighborhood U on which there exist smooth functions $f, g\colon U \to \mathbb{R}$ such that the restriction of X to U is equal to $f \operatorname{grad} g$.

      ii. $\operatorname{curl} X$ is everywhere orthogonal to X.

6. Let G be a compact Lie group. Show that G satisfies the descending chain condition for closed subgroups: If $H_1 \supseteq H_2 \supseteq H_3 \ldots$, with H_i a closed subgroup of G for each i, then there exists n such that $H_k = H_{k+1}$ for all $k \geq n$.

7. Suppose $F\colon S^3 \to S^2$ is a smooth map.

   (a) Show that there exist a smooth 2-form ω on S^2 such that $\int_{S^2} \omega = 1$, and a smooth 1-form on S^3 such that $F^*\omega = d\eta$.

   (b) For any forms ω and η as above, show that $\int_{S^3} \eta \wedge d\eta$ depends only on F, not on the choice of ω or η.

8. Let $\mathrm{O}(n)$ denote the orthogonal group. A *reflection* is a non-identity element $A \in \mathrm{O}(n)$ that fixes every point in some linear $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Let $\mathcal{R}_n \subseteq O(n)$ denote the subset consisting of all reflections. Show that $\mathcal{R}_n$ is a smooth embedded submanifold and is diffeomorphic to the real projective space RP^{n-1}. (Suggestion: It might be useful to consider the action of $\mathrm{O}(n)$ on itself by conjugation.)