Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word “smooth” means $C^\infty$, and all manifolds are assumed to be without boundary.

1. Which 2-manifolds $M$ admit a covering map $\pi : S^2 \to M$?

2. Let $f$ be a smooth real-valued function defined on an open subset $U \subseteq \mathbb{R}^n$. We say $f$ is harmonic if
   \[ \sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2} = 0. \]
   Show that $f$ is harmonic if and only if for every $p \in U$ and every positive number $r$ less than the distance from $p$ to $\partial U$,
   \[ \sum_{i=1}^{n} (-1)^i \int_{S_r(p)} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n = 0, \]
   where $S_r(p)$ is the sphere of radius $r$ around $p$, and $\hat{dx^i}$ indicates that $dx^i$ is omitted from the wedge product.

3. Let $V$ be the following vector field on $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$:
   \[ V = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z}. \]
   (a) Determine the flow of $V$.
   (b) A function $f : M \to \mathbb{R}$ is said to be $V$-invariant if it is invariant under the flow of $V$, or equivalently if it is constant along the integral curves of $V$. Find the $V$-invariant functions.

4. Which (if any) of the following spaces are simply connected?
   (a) The space $N^n$ of $n \times n$ nilpotent matrices over $\mathbb{R}$, $n \geq 1$, with the subspace topology inherited from $\mathbb{R}^{n^2}$. (A square matrix $A$ is nilpotent if $A^k = 0$ for some positive integer $k$.)
   (b) $\mathbb{C}^n \setminus H$, where $H$ is any complex linear subspace of dimension $n - 1$.
   (c) The space $V_2\mathbb{R}^n$ of orthonormal ordered pairs of vectors in $\mathbb{R}^n$, $n \geq 4$, with the subspace topology inherited from $\mathbb{R}^n \times \mathbb{R}^n$. (Suggestion: Note that $V_2\mathbb{R}^n$ can be identified with the space of unit tangent vectors of $S^{n-1}$.)
5. (a) If $\omega$ is a nonvanishing smooth 1-form on a smooth manifold, show that the distribution annihilated by $\omega$ is integrable if and only if $\omega \wedge d\omega = 0$.

(b) If $X$ is a nonvanishing smooth vector field on $\mathbb{R}^3$, prove that the following conditions are equivalent.
   
   i. Every point in $\mathbb{R}^3$ has a neighborhood $U$ on which there exist smooth functions $f, g: U \to \mathbb{R}$ such that the restriction of $X$ to $U$ is equal to $f \text{grad} g$.
   
   ii. $\text{curl} X$ is everywhere orthogonal to $X$.

6. Let $G$ be a compact Lie group. Show that $G$ satisfies the descending chain condition for closed subgroups: If $H_1 \supseteq H_2 \supseteq H_3 \ldots$, with $H_i$ a closed subgroup of $G$ for each $i$, then there exists $n$ such that $H_k = H_{k+1}$ for all $k \geq n$.

7. Suppose $F: S^3 \to S^2$ is a smooth map.

   (a) Show that there exist a smooth 2-form $\omega$ on $S^2$ such that $\int_{S^2} \omega = 1$, and a smooth 1-form on $S^3$ such that $F^* \omega = d\eta$.

   (b) For any forms $\omega$ and $\eta$ as above, show that $\int_{S^3} \eta \wedge d\eta$ depends only on $F$, not on the choice of $\omega$ or $\eta$.

8. Let $O(n)$ denote the orthogonal group. A reflection is a non-identity element $A \in O(n)$ that fixes every point in some linear $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Let $\mathcal{R}_n \subseteq O(n)$ denote the subset consisting of all reflections. Show that $\mathcal{R}_n$ is a smooth embedded submanifold and is diffeomorphic to the real projective space $RP^{n-1}$. (Suggestion: It might be useful to consider the action of $O(n)$ on itself by conjugation.)