Topology and Geometry of Manifolds Preliminary Exam

September 17, 2009

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word "smooth" means C^∞. Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds assumed to be without boundary.

1. Let $A = \{(x,y) \in \mathbb{R}^2 \ : \ y^2 = x^2\}$ and $B = \{(x,y) \in \mathbb{R}^2 \ : \ y^2 = x^2(x+1)\}$. Prove that exactly one of these two plane curves admits a 2-sheeted connected covering, and give such a covering. You should give the covering space explicitly, but may indicate the covering map by pictures and description.

2. For each positive integer n, let S_n be the circle of radius $1/n$ centered at $(1/n, 0)$ and let C be the cone over the union of these circles. (So if we include $\mathbb{R}^2$ into $\mathbb{R}^3$ in the usual way and let $P = (0,0,1)$ be the vertex of the cone, we may represent C as

$$C = \bigcup_{n=1}^{\infty} \{(1-t)Q + tP \in \mathbb{R}^3 \ : \ 0 \le t \le 1, \, Q \in S_n\}.)$$

Prove that the fundamental group of the wedge sum of two copies of C depends on the choice of base points for the wedge sum by showing that

$$\pi_1((C,P) \vee (C,P)) \neq \pi_1((C,O) \vee (C,O)),$$

where O is the origin, (0,0,0).

3. Show that

$$M = \{(Y,Z) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ Y \times Z = (0,0,1)\}$$

is a smoothly embedded submanifold of $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ that is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$, where $\mathbb{S}^1$ is the unit circle. (In $Y \times Z$, the symbol $\times$ indicates the ordinary cross product of vector calculus.)

4. Let $S = \{(x,y,z) \in \mathbb{R}^3 \ : \ 9x^2 + y^2 + z^2 = 9 \text{ and } y + z \ge 0\}$,
and let $\omega = x^2 dx \wedge dy + 3y \, dx \wedge dz + \ln(y+z+1) dy \wedge dz$.
Evaluate $\int_S \omega$, where S is oriented by the normal pointing away from the origin.

5. Let M be a compact manifold with a volume form Ω. Define a bilinear function on the set $C^\infty(M)$ of smooth functions on M by $\langle f,g \rangle = \int_M fg\Omega$. Recall that the divergence $\mathrm{div}(X)$ of a vector field X on M is given by $\mathrm{div}(X)\Omega = \mathcal{L}_X \Omega$. Show that if $\mathrm{div}(X) = 0$, then X is skew-symmetric with respect to $\langle \ , \ \rangle$ as an operator on $C^\infty(M)$: $\langle X(f), g \rangle = -\langle f, X(g) \rangle$.

6. Show that on the 3-sphere $\mathbb{S}^3$, there cannot be three vectors fields, say X_j, $j = 1, 2, 3$, that are linearly independent everywhere and commute: that is, $[X_i, X_j] = 0$ everywhere. Hint: Consider the coframe dual to $\{X_1, X_2, X_3\}$.

   [*It is known that there are three* non-*commuting vector fields on* $\mathbb{S}^3$ *that are linearly independent everywhere, but you need not show this.*]

7. Let ω be a closed differential 2-form on a manifold M. Recall that $(i_X\omega)(Y) = \omega(X, Y)$. The kernel of ω is $\ker\omega = \bigcup_{p\in M} \ker_p \omega$, where

$$\ker_p \omega = \{X \in T_pM \; : \; i_X\omega = 0; \text{ that is, } \omega(X, Y) = 0 \text{ for all } Y \in T_pM\}.$$

   Suppose that the dimension of $\ker_p \omega$ is constant on M; you may use without proof the fact that this implies $\ker\omega$ is an integrable distribution.

   (a) For a smooth function $f : M \to \mathbb{R}$, show that df vanishes on $\ker\omega$ if and only if there is a smooth vector field X on M such that $i_X\omega = df$. (Note that if $\ker\omega$ is nontrivial, X will not be unique.)

   (b) Suppose f is a smooth function on M and $i_X\omega = df$. Show that the Lie derivatives $\mathcal{L}_X f$ and $\mathcal{L}_X\omega$ vanish.

8. Let $\mathbb{G} = G_2(\mathbb{R}^4)$ be the Grassmannian of 2-dimensional linear subspaces of $\mathbb{R}^4$. You may assume that $\mathbb{G}$ is a compact, homogeneous SO(4)-manifold with the obvious action: that is, for $V \in \mathbb{G}$ and $g \in \mathrm{SO}(4)$, let $gV = \{gx \; : \; x \in V\}$. In particular this means $\mathbb{G}$ may be identified with the quotient of SO(4) by the isotropy subgroup I of the plane $\{(x, y, 0, 0) \in \mathbb{R}^4 \; : \; x, y \in \mathbb{R}\}$.

   Let Λ be the 6-dimensional vector space spanned by wedge products $v_1 \wedge v_2$ for $v_1, v_2 \in \mathbb{R}^4$; thus Λ is the vector space of contravariant alternating 2-tensors on $\mathbb{R}^4$. Let $\mathbb{P}$ be the projective space of 1-dimensional subspaces of Λ.

   (a) Define a map $\psi : \mathbb{G} \to \mathbb{P}$ as follows: For each 2-dimensional subspace V of $\mathbb{R}^4$, let $\psi(V) = [v_1 \wedge v_2]$, where $\{v_1, v_2\}$ is any basis for V. Prove ψ is well-defined and is a smooth embedding. Hint: To prove smoothness, use the identification of $\mathbb{G}$ with $\mathrm{SO}(4)/I$.

   [*The map* ψ *is called the "Plücker embedding."*]

   (b) Let $L \subset \mathbb{R}^4$ be a 1-dimensional subspace and $H \subset \mathbb{R}^4$ a 3-dimensional subspace containing L. Define

$$\Sigma = \{V \in \mathbb{G} \; : \; L \subset V \subset H\}.$$

   Show that ψ maps Σ onto a line in $\mathbb{P}$, where a line in $\mathbb{P}$ is the image of a two dimensional subspace of Λ.