Topology and Geometry of Manifolds Preliminary Exam

September 16, 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word "smooth" means C^∞. All manifolds are assumed to be smooth and without boundary unless otherwise specified. All quantities related to manifolds are also assumed to be smooth. (E.g. all maps, vector fields, differential forms, etc. are assumed to be smooth).

(1) Let M be a compact manifold of dimension three, equipped with a 1-form α such that $\alpha \wedge d\alpha$ is a volume form.

Let $f : N \to M$ be an immersion of a 2-dimensional manifold into M. Show that the set

$$A = \{p \in N \,:\, f^*\alpha_p = 0\}$$

has empty interior.

(2) Let $V \subset \mathbb{R}^n$ be a k-dimensional linear subspace of $\mathbb{R}^n$, where $0 \leq k < n$. Find necessary and sufficient conditions for the complement $\mathbb{R}^n \setminus V$ to be simply-connected, and prove your answer.

(3) Let (u, v, x, y) be coordinates on $\mathbb{R}^4$. Show that there are smooth functions, f_1 and f_2, defined in a neighborhood of 0, with $df_1 \wedge df_2 \neq 0$, and satisfying the system of differential equations

$$\frac{\partial f_j}{\partial x} + y\frac{\partial f_j}{\partial u} + x\frac{\partial f_j}{\partial v} = 0\,, \quad \frac{\partial f_j}{\partial y} + x\frac{\partial f_j}{\partial u} + y\frac{\partial f_j}{\partial v} = 0$$

for $j = 1, 2$.

(4) Let X denote the set of pairs of orthogonal lines through the origin in $\mathbb{R}^{n+1}$, viewed as a subspace of the product $\mathbb{P}^n \times \mathbb{P}^n$, where $\mathbb{P}^n$ denotes real projective space. Prove that X is an embedded submanifold of $\mathbb{P}^n \times \mathbb{P}^n$, and determine the dimension of X.

(5) (a) Let M and N be connected, compact, n-dimensional manifolds, and let $f : M \to N$ be a submersion. Show that f is a covering map.

(b) For n a positive integer, let N_n denote the quotient space $S^3/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts on the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 + |z_2|^2\}$ by

$$k \cdot (z_1, z_2) = (e^{(k/n)2\pi i} z_1, e^{(k/n)2\pi i} z_2)\,, \quad k = 0, 1, \ldots, n-1.$$

Suppose $p > 1$ is prime. For what n does there exist a submersion $N_p \to N_n$? Justify your answer.

(6) Recall that the deRham cohomology group $H^k_{dR}(M)$ on a manifold M is the quotient of closed k-forms by exact k-forms. It is a fact that $H^{n-1}_{dR}(\mathbb{R}^n \setminus \{0\})$ has dimension 1. Assume this fact.

(a) Show that the function $H^{n-1}_{dR}(\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ given by

$$[\omega] \mapsto \int_{S^{n-1}} \omega$$

is a well-defined isomorphism. (Here $[\omega]$ denotes the cohomology class represented by the closed form ω.)

(b) Let $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ be a smooth, closed $(n-1)$-form. Suppose that ω is bounded in the sense that there is a constant $c > 0$ such that $|\omega_p(v_1, v_2, \ldots, v_{n-1})| < c$ for all $p \in \mathbb{R}^n \setminus \{0\}$ and all orthonormal sets $\{v_1, \ldots, v_{n-1}\}$ of tangent vectors in $T_p(\mathbb{R}^n \setminus \{0\})$. Show that ω is exact.

(7) Suppose the compact Lie group G acts freely and smoothly on the compact smooth manifold M. Let N denote the set of pairs (x_1, x_2) in $M \times M$ such that x_1 and x_2 lie on the same G-orbit. Show that N is a closed, embedded submanifold of $M \times M$. Show also that the function $f : N \to G$ taking the pair (x_1, x_2) to the unique $g \in G$ such that $gx_1 = x_2$ is smooth.

(8) Let X be a smooth vector field and ω a smooth differential form on the smooth manifold M. Let $L_X\omega$ denote the Lie derivative of ω with respect to X. Show from the definition of Lie derivative that $L_X\omega = 0$ if and only if ω is invariant under the flow of X.