Topology and Geometry of Manifolds Preliminary Exam
September 15, 2011

Do as many of the eight problems as you can. Four problems done correctly will be a clear pass. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word smooth means $C^\infty$, and all manifolds are assumed to be smooth and without boundary unless otherwise specified.

1. Let $X$ be a topological space and suppose that $\pi_1(X, x_0)$ is abelian. We say that the two loops $f, g : (S^1, e) \rightarrow (X, x_0)$ are freely homotopic if there exists a map $H : S^1 \times [0, 1] \rightarrow X$ with $H(z, 0) = f(z)$ and $H(z, 1) = g(z)$ for all $z \in S^1$. Note that $H(\epsilon, t)$ need not be the basepoint $x_0$ for $0 < \epsilon < 1$.

Prove that if $f$ and $g$ are freely homotopic, then they are path homotopic.

2. Let $F(x, t)$ be a smooth real valued function defined in a neighborhood of $(0, 0)$ in $\mathbb{R}^2$. Suppose that $F(0, 0) = F_x(0, 0) = F_t(0, 0) = 0$, and that the quadratic equation

$$F_{xx}(0, 0)w^2 + 2F_{xt}(0, 0)w + F_{tt}(0, 0) = 0$$

has distinct real roots, $w_1 \neq w_2$.

Prove that, inside a possibly smaller neighborhood of $(0, 0)$, the level set $\{(x, t)|F(x, t) = 0\}$ is the union of two smooth curves, $(x_1(t), t)$ and $(x_2(t), t)$, which intersect only at $(0, 0)$.

**Hint:** You may use the fact (which follows from a Taylor expansion with remainder) that

$$f(x) = f(0) + \frac{1}{2}f''(0)w^2$$

with $A(0, 0) = \frac{1}{2}F_{xx}(0, 0)$, $B(0, 0) = \frac{1}{2}F_{xt}(0, 0)$, and $C(0, 0) = \frac{1}{2}F_{tt}(0, 0)$. Look for $x_1(t) = tw_1(t)$ and $x_2(t) = tw_2(t)$.

3. Let $X, Y, Z$ be the vector fields

$$X = \frac{\partial}{\partial y} - \frac{y}{z}\frac{\partial}{\partial z},$$

$$Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial z},$$

$$Z = \frac{\partial}{\partial x} - \frac{x}{y}\frac{\partial}{\partial y}$$

on $\mathbb{R}^3$, and, for $(a, b, c) \in \mathbb{R}^3$, consider the vector field $W = aX + bY + cZ$. Let $\Theta : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the flow of $W$ and let $\Theta_t : \mathbb{R} \rightarrow \mathbb{R}^3$ be defined by $\Theta_t(p) = \Theta(t, p)$.

Prove that $\Theta_t$ is a rotation about the vector $(a, b, c) \in \mathbb{R}^3$; i.e. $\Theta_t \in SO(3)$ and $\Theta_t(a, b, c) = (a, b, c)$. In computing $\Theta_t$, you will see that $W$ is complete.

4. Suppose that $M$ and $N$ are smooth manifolds and that $\pi : M \rightarrow N$ is a surjective smooth submersion with connected fibers. We say that a tangent vector $X \in T_pM$ is **vertical** if $d\pi_p(X) = 0$.

Let $\omega \in \Omega^k(M)$. Show that $\omega = \pi^*\eta$, for some $\eta \in \Omega^k(N)$, if and only if $X \vdash \omega_p = 0$ and $X \vdash d\omega_p = 0$ for every $p \in M$ and every vertical vector $X \in T_pM$.

**Hint:** Start by proving the result in the special case that $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ is a projection onto the first $n$ coordinates.
5. Let \((M^n, g)\) denote an oriented Riemannian manifold. For each tangent vector \(X_p \in T_p(M)\) we define a cotangent vector \(\omega^{X_p} \in \Omega^1_p(M)\) by requiring that

\[
\omega^{X_p} = (X_p, \cdot)_g
\]

(a) Show that the formula \((\omega^{X_p}, \omega^{Y_p})^g := ((X_p, Y_p)_g\) defines an inner product on the entire vector space \(\Omega^1_p(M)\) (i.e. that every \(\omega \in \Omega^1_p(M) = \omega^{X_p}\) for a unique \(X_p\), and that \((\omega_1, \omega_2)^g\) is bilinear and positive).

(b) Let \(\{\omega_j\}_{j=1\ldots n}\) be a positively oriented orthonormal frame for \(\Omega^1_p(M)\), and define an inner product on \(\Omega^k_p(M)\) by declaring that the wedge products \(\{\omega_{i_1} \wedge \omega_{i_2} \wedge \ldots \wedge \omega_{i_k}\}_{1 \leq i_1 < i_2 < \ldots < i_k \leq n}\) form an orthonormal basis for \(\Omega^k_p(M)\). The volume form \(V^g \in \Omega^n(M)\) is then the unique \(n\)-form satisfying, at every \(p\),

\[
V^g_p := \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n
\]

For this problem, you should simply accept this fact that these definitions are independent of the choice of orthonormal frame.

For a pair of vector fields, \(X\) and \(Y\), we define an \((n-2)\)-form

\[
\eta^{X,Y} := V^g(X,Y,\ldots)
\]

Prove that, for every \(\theta \in \Omega^2(M)\),

\[
\eta^{X,Y} \wedge \theta = (\omega^{X} \wedge \omega^{Y}, \theta)^g V^g
\]

(c) Prove that, if \(X\) and \(Y\) are linearly independent vector fields satisfying \([X,Y] = 0\), then there is a 1-form \(\theta\) such that \(d\eta^{X,Y} = \theta \wedge \eta^{X,Y}\). This part of the problem does not depend on the previous parts.

6. Suppose that \(G\) and \(H\) are Lie groups, that \(G\) is connected, and that \(\Phi : G \to H\) is a smooth map sending the identity of \(G\) to the identity of \(H\), with the property that, for each left invariant vector field \(X\), there exists a (necessarily unique) left invariant vector field \(Y\) on \(H\) which is \(\Phi\)-related to \(X\).

(a) Prove that \(\Phi\) induces a map \(\phi : \mathfrak{g} \to \mathfrak{h}\) of Lie algebras, where \(\mathfrak{g}\) is the Lie algebra of \(G\) and \(\mathfrak{h}\) is the Lie algebra of \(H\).

(b) Prove that \(\Phi\) is a Lie group homomorphism. Hint: Prove that the graph of \(\Phi\) is a Lie subgroup of \(G \times H\).

What must the Lie algebra be?

7. Let \(F\) be a smooth mapping from a compact oriented \(d\)-dimensional manifold \(M\) to itself. Suppose that \(m \in M\) is a regular point of \(F\) and its inverse image contains exactly \(N\) points. Let \(\omega \in \Omega^d(M)\) satisfy \(\int_M \omega \neq 0\). Show that

\[
\left| \frac{\int_M F^*\omega}{\int_M \omega} \right| \leq N
\]

You may assume the fact that a \(d\)-form on a \(d\)-dimensional manifold which has integral zero is exact. You should prove any facts you use about the degree of a map.

8. Let \(p : E \to B\) be a finite sheeted smooth covering map (with \(E\) connected), and let \(G\) denote the group of deck transformations of \(p\). Suppose that \(G\) acts transitively on each fiber. The goal of this problem is to prove that \(p^* : H^*_dR(B) \to (H^*_dR(E))^G\) is an isomorphism, where \((H^*_dR(E))^G\) denotes the \(G\)-fixed points of \(H^*_dR(E)\).

(a) If \(\omega \in \Omega^k E\), define \(\text{tr}(\omega) \in \Omega^k B\) by

\[
\text{tr}(\omega)(x_1, \ldots, x_k) = \sum_{x \in p^{-1}(b)} \omega_x((dp_x)^{-1}x_1, \ldots, (dp_x)^{-1}x_k)
\]

Prove that \(\text{tr}(\omega)\) is, in fact, smooth, and that \(d\text{tr}(\omega) = \text{tr}(d\omega)\).

(b) Use part a to prove that \(p^* : H^*_dR(B) \to (H^*_dR(E))^G\) is one-to-one.

(c) Prove that \(p^* : H^*_dR(B) \to (H^*_dR(E))^G\) is onto. Hint: What is \(p^*(\text{tr}(\omega))\)?