Topology and Geometry of Manifolds Preliminary Exam

September 13, 2012

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word "smooth" means C^∞. Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds assumed to be without boundary.

1. Define an equivalence relation on the closed unit ball in $\mathbb{R}^3$ by identifying p and $-p$ for every p in the boundary, and let X be the resulting quotient space. Find a presentation of the fundamental group, and give a specific loop representing each generator.

2. Let M be the connected sum of two 2-tori: $M = \mathbb{T}^2 \# \mathbb{T}^2$. (Thus M is the orientable surface of genus 2.) Show that M admits a (connnected) two-sheeted covering space.

3. Let f be a continuous real-valued function on a manifold M such that $f(x) > 0$ for all $x \in M$. Show that there is a smooth real-valued function g on M such that $f(x) > g(x) > 0$ for all $x \in M$. *Note that the given function f is only assumed to be continuous, but the desired function g must be smooth.*

4. Let G be a Lie group. Show that there is a neighborhood N of the identity $e \in G$ which contains no subgroup of G apart from $\{e\}$.

5. Suppose that D and E are involutive distributions on the n-manifold M such that $T_pM = D_p \oplus E_p$ for all $p \in M$, and let d be the dimension of D. Show that for each $p \in M$ there is a coordinate chart in a neighborhood U of p such that D is spanned by the first d coordinate partial derivatives in this chart while E is spanned by the remaining partial derivatives, at every point of U.

6. Let (M, g) be a compact, oriented, Riemannian manifold with boundary, let dV be the Riemannian volume form on M, and let f and X be a smooth function and a smooth vector field, respectively, on M. Recall that the divergence $\text{div}(X)$ is determined by the equation $d(i_X dV) = (\text{div } X)dV$.

   (a) Show that $\text{div}(fX) = f \,\text{div } X + g(\text{grad} f, X)$.

   (b) On ∂M, let N be the outward unit normal vector field and let dU be the induced Riemannian volume form. Prove that

   $$\int_M g(\text{grad} f, X)\, dV = \int_{\partial M} f\, g(X, N)\, dU - \int_M (f \,\text{div } X)\, dV.$$

   (c) Show that the standard integration by parts formula of one-variable calculus is a special case of (b).

7. Suppose M is a compact, $2n$-dimensional manifold that admits a symplectic structure. The latter means that there is a closed differential two-form ω such that $\omega^n = \omega \wedge ... \wedge \omega$ is nonzero everywhere on M.

   (a) Show that the even-dimensional deRham cohomology groups $H^{2k}_{dR}(M)$, $k = 1, ..., n$, are nontrivial.

   of positive even dimension

   (b) Show that the only sphere that admits a symplectic structure is the 2-dimensional sphere $\mathbb{S}^2$.

8. Let $M = \mathbb{R}^n \backslash \{0\}$ and let $||x||$ be the Euclidean norm of $(x^1, x^2, ..., x^n) \in M$. Consider the vector field

$$X = \frac{1}{||x||^n} \sum_{j=1}^{n} x^j \frac{\partial}{\partial x^j}$$

   on M.

   (a) Find the flow of X. Is X complete?

   (b) Show that the $(n-1)$-form $\omega = i_X(dx^1 \wedge dx^2 \wedge ... \wedge dx^n)$ is closed and use this fact to compute $\mathcal{L}_X(dx^1 \wedge dx^2 \wedge ... \wedge dx^n)$.