TOPOLOGY AND GEOMETRY OF MANIFOLDS
PRELIMINARY EXAM

September 12, 2013

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

(1) Let $n \geq 2$. Identify with proof the fundamental group of the following submanifold of $\mathbb{R}^n \times \mathbb{R}^n$:

$$M_n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \, : \, x \neq y\} \, .$$

(2) Let $n \geq 1$. Recall that complex projective space $\mathbb{CP}^n$ is the smooth manifold obtained as the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $z \sim w$ if there is $\lambda \in \mathbb{C} \setminus \{0\}$ such that $z = \lambda w$.

(a) Exhibit explicitly a smooth atlas for $\mathbb{CP}^n$. You do not have to prove that the charts in your atlas are homeomorphisms, but you should define the atlas clearly and you should verify that any two charts in your atlas are smoothly compatible.

(b) Show (by whatever reasoning you prefer) that $\mathbb{CP}^n$ is compact and connected.

(3) Let (x, y, z) denote the standard coordinates on $\mathbb{R}^3$.

(a) Show that the three vector fields

$$X = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}\,, \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\,, \quad Z = \frac{\partial}{\partial z}$$

on $\mathbb{R}^3$ form a basis for a 3-dimensional Lie subalgebra of the Lie algebra of smooth vector fields on $\mathbb{R}^3$.

(b) Derive the multiplication law $(x, y, z) \cdot (x', y', z')$ for a group structure on $\mathbb{R}^3$ with respect to which $\mathbb{R}^3$ becomes a Lie group whose left invariant vector fields are $\mathrm{span}_{\mathbb{R}} \{X, Y, Z\}$.

Hint: Compute the flow of the vector field $aX + bY + cZ$ for $(a, b, c) \in \mathbb{R}^3$.

(4) Show that every closed 1-form on a compact, simply connected smooth manifold vanishes at at least one point.

(5) Let $f : M \to \mathbb{R}$ be a smooth real-valued function on a smooth manifold M, and let $p \in M$ be a critical point of f. The critical point p is called a *nondegenerate* critical point if the matrix of second derivatives of f at p, with respect to a local coordinate system, is nonsingular.

(a) Prove that this notion of nondegeneracy is independent of the choice of local coordinate system.
(b) Give an explicit example of a smooth function $S^1 \times S^1 \to \mathbb{R}$ such that all of its critical points are nondegenerate, and show that your example has this property.

(6) Consider the one form $\alpha = dw + ydx + zdy + wdz$ on $\mathbb{R}^4$, where (w, x, y, z) are the standard coordinates. Show that there does not exist an embedding $F : \mathbb{R}^3 \to \mathbb{R}^4$ such that $F^*\alpha = 0$.

(7) Let M be the set of ordered pairs of non-parallel lines in $\mathbb{R}^2$.

(a) Use the appropriate theorems about group actions and homogeneous spaces to show that M has a natural topology and smooth structure making it into a homogeneous space, and identify explicitly a quotient of Lie groups which is diffeomorphic to M. Be sure to state and carefully verify the hypotheses of the theorems you use.
(b) Determine, with proof, whether or not M is connected.

Hint: Let $G \subset GL(3, \mathbb{R})$ be the set of 3×3 matrices of the form

$$\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}, \quad A \in GL(2, \mathbb{R}), \quad a \in \mathbb{R}^2.$$

Show that G is a Lie group and consider an action of G as affine transformations of $\mathbb{R}^2$.

(8) Let X be a connected topological manifold and let G be a finite group with the discrete topology. Let

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

be a continuous effective action of G on X. (Recall that a group action $G \times X \to X$ is *effective* if for each $g \in G$ such that $g \neq e$, there is $x \in X$ with $g \cdot x \neq x$.) Prove that the projection map $X \to X/G$ is a covering map if and only if the action is free, that is, if and only if $g \in G$ and $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

For both directions, please provide the details of a proof. That is, you are not allowed to just say that either direction is a special case of some theorem you might know.