Topology and Geometry of Manifolds
Preliminary Exam

September 11, 2014

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word "smooth" means C^∞. Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds assumed to be without boundary.

(1) Let G be a topological group with identity e. Let $*$ denote path multiplication as well as the multiplication on $\pi_1(G,e)$; i.e., if ω, η are loops in G based at e, then $[\omega] * [\eta] = [\omega * \eta] \in \pi_1(G,e)$. Prove that

$$[\omega] * [\eta] = [\omega\eta],$$

where $(\omega\eta)(t) = \omega(t)\eta(t)$, the multiplication in G of $\omega(t)$ and $\eta(t)$.

(2) Let M be a smooth manifold and $A \subset U \subset M$ with A a closed set and U an open set. Prove that if $f : A \longrightarrow \mathbb{R}$ is a smooth function, then there exists a smooth function $\tilde{f} : M \longrightarrow \mathbb{R}$ such that $\tilde{f}|_A = f$ and $\operatorname{supp}(\tilde{f}) \subset U$.

(3) Suppose M is a compact nonempty manifold of dimension n, $n > 0$, and $f : M \longrightarrow \mathbb{R}^n$ is a smooth map. Show that f is not an immersion.

(4) Let G be a Lie group with identity e. Prove that there exists a neighborhood U of e such that each element of U has a unique square root in U (i.e. prove that for each $x \in U$ there exists a unique $v \in U$ with $v^2 = x$).

Hint: You will likely need to make use of properties of the exponential map.

(5) Let $SL(n, \mathbb{R})$ be the Lie subgroup of $GL(n, \mathbb{R})$ consisting of all $n \times n$ matrices of determinant 1. Prove that the Lie algebra of $SL(n, \mathbb{R})$ is isomorphic to the Lie algebra of all $n \times n$ matrices of trace 0 with Lie bracket given by $[A, B] = AB - BA$. (You may use whatever facts you know about the Lie algebra of $GL(n, \mathbb{R})$.)

(6) Let

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

be vector fields on $\mathbb{R}^3$.

Does there exist a neighborhood U of the origin, and coordinates

$$\phi = (x_1, x_2, x_3) : U \longrightarrow \mathbb{R}^3,$$

such that

$$X|_U = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad Y|_U = \frac{\partial}{\partial x_2} \text{ ?}$$

(Either prove the existence of such coordinates or demonstrate that no such coordinates exist.)

(7) Show that there exists a smooth manifold M, a point $x_0 \in M$ and $\gamma : [0,1] \longrightarrow M$ a smooth curve with $\gamma(0) = \gamma(1) = x_0$ such that $\int_\gamma \omega = 0$ for every smooth closed one-form ω on M, but with $[\gamma] \in \pi_1(M, x_0)$ not the identity element.

(8) Let M be an n-dimensional smooth manifold. For each $t \in [0,1]$, consider the map

$$i_t : M \longrightarrow M \times [0,1] \qquad \text{given by} \qquad i_t(m) = (m, t).$$

Let $\pi_M : M \times [0,1] \longrightarrow M$ denote the projection. If ω is a k-form on $M \times [0,1]$, then ω can be written uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where $\omega_1 \in \Omega^k(M \times [0,1])$ is a k-form on $M \times [0,1]$ and $\eta \in \Omega^{k-1}(M \times [0,1])$ is a $k-1$-form on $M \times [0,1]$ such that $X \lrcorner\, \omega_1 = 0$ and $X \lrcorner\, \eta = 0$ if $X \in \mathrm{Ker}(\pi_M)_*$ (you do not have to prove this fact). Define

$$G : \Omega^k(M \times [0,1]) \longrightarrow \Omega^{k-1}(M)$$

by

$$G(\omega)_p(v_1, \ldots, v_{k-1}) = \int_0^1 \eta_{(p,t)}(i_{t*}v_1, \ldots, i_{t*}v_{k-1})dt.$$

Prove that

$$dG(\omega) + G(d\omega) = i_1^*\omega - i_0^*\omega.$$

(Some of you may recognize that this is a key step in proving that smoothly homotopic maps induce the same map in deRham cohomology.)