Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word “smooth” means \( C^\infty \). Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds assumed to be without boundary.

(1) Let \( G \) be a topological group with identity \( e \). Let \( * \) denote path multiplication as well as the multiplication on \( \pi_1(G, e) \); i.e., if \( \omega, \eta \) are loops in \( G \) based at \( e \), then \( [\omega] * [\eta] = [\omega * \eta] \in \pi_1(G, e) \). Prove that

\[
[\omega] * [\eta] = [\omega \eta],
\]

where \( (\omega \eta)(t) = \omega(t) \eta(t) \), the multiplication in \( G \) of \( \omega(t) \) and \( \eta(t) \).

(2) Let \( M \) be a smooth manifold and \( A \subset U \subset M \) with \( A \) a closed set and \( U \) an open set. Prove that if \( f : A \rightarrow \mathbb{R} \) is a smooth function, then there exists a smooth function \( \tilde{f} : M \rightarrow \mathbb{R} \) such that \( \tilde{f}|_A = f \) and \( \text{supp}(\tilde{f}) \subset U \).

(3) Suppose \( M \) is a compact nonempty manifold of dimension \( n, n > 0 \), and \( f : M \rightarrow \mathbb{R}^n \) is a smooth map. Show that \( f \) is not an immersion.

(4) Let \( G \) be a Lie group with identity \( e \). Prove that there exists a neighborhood \( U \) of \( e \) such that each element of \( U \) has a unique square root in \( U \) (i.e. prove that for each \( x \in U \) there exists a unique \( v \in U \) with \( v^2 = x \)).

**Hint:** You will likely need to make use of properties of the exponential map.
(5) Let $SL(n, \mathbb{R})$ be the Lie subgroup of $GL(n, \mathbb{R})$ consisting of all $n \times n$ matrices of determinant 1. Prove that the Lie algebra of $SL(n, \mathbb{R})$ is isomorphic to the Lie algebra of all $n \times n$ matrices of trace 0 with Lie bracket given by $[A, B] = AB - BA$. (You may use whatever facts you know about the Lie algebra of $GL(n, \mathbb{R})$.)

(6) Let

$$X = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \quad \text{and} \quad Y = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

be vector fields on $\mathbb{R}^3$.

Does there exist a neighborhood $U$ of the origin, and coordinates

$$\phi = (x_1, x_2, x_3) : U \rightarrow \mathbb{R}^3,$$

such that

$$X|_U = \frac{\partial}{\partial x_1} \quad \text{and} \quad Y|_U = \frac{\partial}{\partial x_2} ?$$

(Either prove the existence of such coordinates or demonstrate that no such coordinates exist.)

(7) Show that there exists a smooth manifold $M$, a point $x_0 \in M$ and $\gamma : [0, 1] \rightarrow M$ a smooth curve with $\gamma(0) = \gamma(1) = x_0$ such that $\int_{\gamma} \omega = 0$ for every smooth one-form $\omega$ on $M$, but with $[\gamma] \in \pi_1(M, x_0)$ not the identity element. (Some of you may recognize that this is a key step in proving that smoothly homotopic maps induce the same map in deRham cohomology.)

(8) Let $M$ be an $n$-dimensional manifold. For each $t \in [0, 1]$, consider the map

$$i_t : M \rightarrow M \times [0, 1] \quad \text{given by} \quad i_t(m) = (m, t).$$

Let $\pi_M : M \times [0, 1] \rightarrow M$ denote the projection. If $\omega$ is a $k$-form on $M \times [0, 1]$, then $\omega$ can be written uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where $\omega_1 \in \Omega^k(M \times [0, 1])$ is a $k$-form on $M \times [0, 1]$ and $\eta \in \Omega^{k-1}(M \times [0, 1])$ is a $k-1$-form on $M \times [0, 1]$ such that $X_\gamma \omega_1 = 0$ and $X_\gamma \eta = 0$ if $X \in \text{Ker}(\pi_M)_*$ (you do not have to prove this fact). Define

$$G : \Omega^i(M \times [0, 1]) \rightarrow \Omega^{i-1}(M)$$

by

$$G(\omega)_p(v_1, \ldots, v_{k-1}) = \int_0^1 \eta(p, t)(i_{t*}v_1, \ldots, i_{t*}v_{k-1}) dt.$$

Prove that

$$dG(\omega) + G(d\omega) = i^*_t \omega - i^*_0 \omega.$$

(Some of you may recognize that this is a key step in proving that smoothly homotopic maps induce the same map in deRham cohomology.)