Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word “smooth” means $C^\infty$. Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds are assumed to be without boundary. Subsets of $\mathbb{R}^n$ are assumed to have the Euclidean topology, and $\mathbb{R}^n$ and $S^n$ are assumed to have their standard smooth structures.

1. Let $M$ be the quotient space obtained from the standard unit sphere $S^2 \subset \mathbb{R}^3$ by identifying the north and south poles $(0,0,\pm 1)$ of $S^2$. Find a presentation for the fundamental group $\pi_1(M,p)$, where $p$ is the identified poles, giving a specific loop for each generator.

2. Let $S^2$ denote the unit sphere in $\mathbb{R}^3$ and let $(x,y,z)$ be the standard coordinates on $\mathbb{R}^3$. Show that the map
   
   \[ f : S^2 \to \mathbb{R}^3 : (x,y,z) \mapsto ((1 - z^2)x, (1 - z^2)y, z) \]

   is a topological embedding but not a smooth embedding.

3. Let $M$ be a smooth manifold, and let $\Phi = (\Phi^1, ..., \Phi^k) : M \to \mathbb{R}^k$ and $f : M \to \mathbb{R}$ be smooth functions. Suppose $C$ is a regular level set of $\Phi$ and $p \in C$ is a point where $f$ attains a minimum or maximum value on $C$. Prove that there exist real numbers $\lambda_1, ..., \lambda_k$ such that
   \[ df_p = \lambda_1 d\Phi_p^1 + ... + \lambda_k d\Phi_p^k. \]

   Remark: The numbers $\lambda_1, ..., \lambda_k$ are called Lagrange multipliers.

4. Let $M \subset \mathbb{R}^3$ be a compact, 3-dimensional smooth manifold with boundary, and assume that the origin is in the interior of $M$. Give the boundary $\partial M$ of $M$ the induced (Stokes) orientation. Compute $\int_{\partial M} \omega$, where $\omega$ is the form
   \[ \omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}. \]

5. Let $M$ be a compact, connected, orientable smooth manifold. Let $N$ be the quotient space obtained from the equivalence relation generated by $\phi(p) \sim p$ for all $p \in M$, where $\phi : M \to M$ is a smooth diffeomorphism satisfying the following conditions:

   (i) $\phi$ is orientation reversing;
   (ii) $\phi(p) \neq p$ for all $p \in M$; and
   (iii) $\phi(\phi(p)) = p$ for all $p \in M$.

   Prove that $N$ can be given the structure of a nonorientable smooth manifold with the quotient topology.
6. Let $\Omega = dx \wedge dy \wedge du \wedge dv$ denote the standard volume form on $\mathbb{R}^4$ and let $M \subset \mathbb{R}^4$ be a smooth, compact, 4-dimensional submanifold of $\mathbb{R}^4$ with smooth boundary $\partial M$.

Let $X$ denote the vector field on $\mathbb{R}^4$ given by

$$X = y^4 \frac{\partial}{\partial x} - x^4 \frac{\partial}{\partial y} + u^4 \frac{\partial}{\partial u} - v^4 \frac{\partial}{\partial v},$$

let $\nu_t : \mathbb{R}^4 \to \mathbb{R}^4$ denote its flow, and let $M_t = \nu_t(M)$.

Show that $\int_{M_t} \Omega$ is constant. (Note: You may assume without proof that $X$ is a complete vector field.)

7. Let $f : S^3 \to S^2$ be a submersion, where $S^n$ denotes the $n$-sphere. Prove that $f$ is surjective but has no section.

(Recall that if $f : M \to N$ is a surjective submersion, a section of $f$ is a smooth map $g : N \to M$ such that $f \circ g = Id_N$.)

8. Suppose $\omega$ is a closed two-form on a smooth manifold $M$ such that the kernel $K_p$ of $\omega$ at $p \in M$,

$$K_p = \{ V \in T_pM : \omega(V, W) = 0 \text{ for all } W \in T_pM \},$$

has constant, positive dimension. A submanifold $N$ of $M$ is said to be characteristic if $T_pN = K_p$ at every point $p \in N$. Prove that there is a foliation of $M$ by characteristic submanifolds.