Topology and Geometry of Manifolds Preliminary Exam

September 17, 2015

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word "smooth" means C^∞. Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds are assumed to be without boundary. Subsets of $\mathbb{R}^n$ are assumed to have the Euclidean topology, and $\mathbb{R}^n$ and $\mathbb{S}^n$ are assumed to have their standard smooth structures.

1. Let M be the quotient space obtained from the standard unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ by identifying the north and south poles $(0, 0, \pm 1)$ of $\mathbb{S}^2$. Find a presentation for the fundamental group $\pi_1(M, p)$, where p is the identified poles, giving a specific loop for each generator.

2. Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$ and let (x, y, z) be the standard coordinates on $\mathbb{R}^3$. Show that the map

$$f : \mathbb{S}^2 \to \mathbb{R}^3 \ : \ (x, y, z) \mapsto \big((1 - z^2)x, (1 - z^2)y, z\big)$$

   is a topological embedding but not a smooth embedding.

3. Let M be a smooth manifold, and let $\Phi = (\Phi^1, ..., \Phi^k) : M \to \mathbb{R}^k$ and $f : M \to \mathbb{R}$ be smooth functions. Suppose C is a regular level set of Φ and $p \in C$ is a point where f attains a minimum or maximum value on C. Prove that there exist real numbers λ_1, ..., λ_k such that

$$df_p = \lambda_1 d\Phi^1_p + ... + \lambda_k d\Phi^k_p \, .$$

   Remark: The numbers λ_1, ..., λ_k are called *Lagrange multipliers*.

4. Let $M \subset \mathbb{R}^3$ be a compact, 3-dimensional smooth manifold with boundary, and assume that the origin is in the interior of M. Give the boundary ∂M of M the induced (Stokes) orientation. Compute $\int_{\partial M} \omega$, where ω is the form

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \, .$$

5. Let M be a compact, connected, orientable smooth manifold. Let N be the quotient space obtained from the equivalence relation generated by $\phi(p) \sim p$ for all $p \in M$, where $\phi : M \to M$ is a smooth diffeomorphism satisfying the following conditions:

   (i) ϕ is orientation reversing;

   (ii) $\phi(p) \neq p$ for all $p \in M$; and

   (iii) $\phi(\phi(p)) = p$ for all $p \in M$.

   Prove that N can be given the structure of a nonorientable smooth manifold with the quotient topology.

6. Let $\Omega = dx \wedge dy \wedge du \wedge dv$ denote the standard volume form on $\mathbb{R}^4$ and let $M \subset \mathbb{R}^4$ be a smooth, compact, 4-dimensional submanifold of $\mathbb{R}^4$ with smooth boundary ∂M.

   Let X denote the vector field on $\mathbb{R}^4$ given by

$$X = y^4 \frac{\partial}{\partial x} - x^4 \frac{\partial}{\partial y} + v^4 \frac{\partial}{\partial u} - u^4 \frac{\partial}{\partial v}\,,$$

   let $\nu_t : \mathbb{R}^4 \to \mathbb{R}^4$ denote its flow, and let $M_t = \nu_t(M)$.

   Show that $\displaystyle\int_{M_t} \Omega$ is constant. (Note: You may assume without proof that X is a complete vector field.)

7. Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a submersion, where $\mathbb{S}^n$ denotes the n-sphere. Prove that f is surjective but has no section.

   (Recall that if $f : M \to N$ is a surjective submersion, a *section* of f is a smooth map $g : N \to M$ such that $f \circ g = Id_N$.)

8. Suppose ω is a closed two-form on a smooth manifold M such that the kernel K_p of ω at $p \in M$,

$$K_p = \{V \in T_pM \ : \ \omega(V, W) = 0 \text{ for all } W \in T_pM\}\,,$$

   has constant, positive dimension. A submanifold N of M is said to be *characteristic* if $T_pN = K_p$ at every point $p \in N$. Prove that there is a foliation of M by characteristic submanifolds.