Topology and Geometry of Manifolds Preliminary Exam

September 15, 2016

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word smooth means C^∞ . Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds are assumed to be without boundary. Subsets of $\mathbb{R}^n$ are assumed to have the Euclidean topology, and $\mathbb{R}^n$ is assumed to have its standard smooth structure.

(1) Suppose that M is an n-manifold embedded in $\mathbb{R}^{n+1}$. Prove that M is locally the graph of a real-valued function of n variables. More precisely, let $x = (x^1, x^2, \ldots, x^{n+1})$ denote the standard coordinates on $\mathbb{R}^{n+1}$ and let $\hat{x}_k \in \mathbb{R}^n$ denote the point obtained from x by removing the k-th coordinate. Show that for any point $p \in M \subset \mathbb{R}^{n+1}$, there is an integer k and a real-valued function f defined in an open neighborhood U of $\hat{p}_k$ such that the set

$$\{(x^1, \ldots, x^{k-1}, f(\hat{x}_k), x^{k+1}, \ldots, x^{n+1}) \mid \hat{x}_k \in U\}$$

is an open neighborhood of p in M.

(2) (a) Let $U \subset \mathbb{R}^2$ be simply connected with x in the interior of U. Prove that the abelianization of $\pi_1(U \setminus \{x\})$ is isomorphic to $\mathbb{Z}$.

(b) Use part (a) to show that the union of the xy-plane with the xz-plane in $\mathbb{R}^3$ is not a topological manifold of dimension 2. (This also follows from Invariance of Domain, but you will only receive partial credit if you use it.)

(3) Let M be a simply connected smooth manifold and D a 1-dimensional distribution on M. Prove that there exists a vector field X on M such that X_p spans D_p for each $p \in M$.

(4) Let $f : P \to M$ be a smooth map from a compact, oriented, simply connected, 3-dimensional manifold to a compact, oriented, connected 2-dimensional manifold. (By the Poincaré Conjecture, $P = S^3$, but you will not need to use this.) Let ω be a 2-form on M with $\displaystyle\int_M \omega = 1$. One can show that there is a 1-form η such that $f^*\omega = d\eta$.

Show that the number $\displaystyle\int_P \eta \wedge f^*\omega$ is independent of the choices of ω and η.

(5) Let M denote the set of unoriented triangles in $\mathbb{R}^3$ with one vertex at the origin. Find a transitive action of a Lie group G on M and use it to identify M with a homogeneous space G/H. Show that this implies that M is naturally a connected, smooth manifold, and compute its dimension.

(over)

(6) Let $G = \{A \in GL(n,\mathbb{R}) \,|\, A \cdot A^t = cI\ ,\ c \in \mathbb{R}\}$, where I denotes the identity matrix. Show that G is a closed Lie subgroup of $GL(n,\mathbb{R})$. Compute the Lie algebra of G.

(7) Let (u, v, x, y) be the standard coordinates on $\mathbb{R}^4$. Show that there are functions $f_1(u, v, x, y)$ and $f_2(u, v, x, y)$ defined on a neighborhood of $(0, 0, 0, 0)$ such that $df_1 \wedge df_2$ never vanishes, satisfying the following system of partial differential equations:

$$(1 - uv)\frac{\partial f_j}{\partial u} - y\frac{\partial f_j}{\partial x} + vy\frac{\partial f_j}{\partial y} = 0$$
$$(1 - uv)\frac{\partial f_j}{\partial v} + ux\frac{\partial f_j}{\partial x} - x\frac{\partial f_j}{\partial y} = 0$$

$j = 1, 2$.

(8) Let X be a complete vector field on the manifold M and let $\nu_t : M \to M$, $t \in \mathbb{R}$, be its flow.

(a) Show that the family of maps $d\nu_t : TM \to TM$, $t \in \mathbb{R}$, is the flow of a complete vector field, $\overline{X}$, on the manifold TM.

(b) Let $X = \sum_{i=1}^{n} X^i(x)\frac{\partial}{\partial x^i}$, where $x = (x^1, x^2, \ldots, x^n)$ are local coordinates on M. Suppose that

$$\overline{X} = \sum_{i=1}^{n} A^i(x, \dot{x})\frac{\partial}{\partial x^i} + \sum_{i=1}^{n} B^i(x, \dot{x})\frac{\partial}{\partial \dot{x}^i}$$

where $(x, \dot{x}) = (x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n)$ denote the induced coordinates on TM. Find expressions for A and B.