REAL ANALYSIS PRELIMINARY EXAM

September 2005

Instructions: Do as many of the following eight problems as you can. The problems are all weighted equally, but in problems with more than one part, the parts may not be weighted equally. Four problems completely correct will be considered a clear pass. You may use any standard theorem, identifying it either by name or by quoting it in full; be sure to verify its hypotheses.

In problems dealing with functions on (subsets of) $\mathbb{R}^n$, the measure in question is always Lebesgue measure.

1. Prove that there exists a nonnegative continuous function on $[0,1]$ such that $f(0) = f(1) = 0$, and for a.e. $x \in [0,1]$ the derivative $f'(x)$ exists and is strictly positive.

2. Let l^p $(1 \le p < \infty)$ and l^∞ be the spaces of p-th power summable functions and bounded functions, respectively, on the positive integers.

   (a) Suppose $1 < p < \infty$. Show that $f_n \to f$ weakly in l^p (that is, $\sum_{k=1}^\infty f_n(k)g(k) \to \sum_{k=1}^\infty f(k)g(k)$ for all $g \in l^q$, where $p^{-1} + q^{-1} = 1$) $\iff$ $f_n \to f$ pointwise and $\sup_n \|f_n\|_p < \infty$.

   (b) Is the $\Rightarrow$ implication in part (a) true for $p = 1$? Is the $\Leftarrow$ implication? Justify your answer.

3. For $B \subset \mathbb{R}$, let $\text{Int}(B)$ denote the interior of B and let $\overline{B}$ denote the closure of B (in the usual topology). Given $A \subset \mathbb{R}$, define a sequence $\{A_k\}$ of sets inductively by $A_1 = A$, and for $k \ge 1$, $A_{2k} = \overline{A_{2k-1}}$ and $A_{2k+1} = \text{Int}(A_{2k})$.

   (a) Find $A \subset \mathbb{R}$ such that the family $\{A_k\}_{k\ge1}$ contains four distinct sets.

   (b) Prove that for any $A \subset \mathbb{R}$, the family $\{A_k\}_{k\ge1}$ contains at most four distinct sets.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exists $k > 0$ such that for every $y \in \mathbb{R}$ there are at most k distinct $x \in \mathbb{R}$ with $f(x) = y$. Prove that the derivative $f'(x)$ exists for a.e. $x \in \mathbb{R}$.

5. Suppose $f \in L^p(\mathbb{R})$, where $1 < p < \infty$.

   (a) Show that f is integrable over every finite interval $[a,\, a+h]$ and that

   $$\left| \int_a^{a+h} f(x)\,dx \right|^p \leq h^{p-1} \int_a^{a+h} |f(x)|^p\,dx.$$

   (b) Suppose there is a constant $C < 1$ such that for all $a \in \mathbb{R}$ and $h > 0$,

   $$\left| \int_a^{a+h} f(x)\,dx \right|^p \leq C h^{p-1} \int_a^{a+h} |f(x)|^p\,dx.$$

   Show that $f = 0$ a.e. (Hint: Divide by h^p.)

6. Suppose $f \in L^1(\mathbb{R}^3)$. Show that the integral

   $$\phi(x) = \int_{\mathbb{R}^3} \frac{f(y)\,dy}{|x-y|}$$

   converges for a.e. $x \in \mathbb{R}^3$ and that the resulting function ϕ is Lebesgue integrable over any bounded measurable set in $\mathbb{R}^3$.

7. In this problem you can use the fact that simple functions (finite linear combinations of characteristic functions of measurable sets) are dense in $L^2([0,1])$.

   (a) Show that $C([0,1])$ is dense in $L^2([0,1])$ (in the L^2 norm).

   (b) Show that there is an orthonormal basis $\{f_n\}_{n=0}^{\infty}$ for $L^2([0,1])$ such that f_n is a polynomial of degree n for all n.

8. Show that for all nonnegative measurable functions f and g on $[0,\infty)$,

   $$\int_0^\infty \int_0^x \frac{f(t)g(x)}{x}\,dt\,dx \leq 2\left[\int_0^\infty f(t)^2\,dt \int_0^\infty g(x)^2\,dx\right]^{1/2}.$$

   (Hint: Write $\dfrac{f(t)g(x)}{x} = \dfrac{f(t)t^{1/4}}{x^{3/4}} \cdot \dfrac{g(x)}{t^{1/4}x^{1/4}}$.)
   Conclude that the operator T defined by $Th(x) = x^{-1}\int_0^x h(t)\,dt$ is bounded on $L^2([0,\infty))$.