Real Analysis Prelim problems 2006

- In answering the questions on this exam, you may use any standard theorem from your real analysis course. You must, however, state the result that you are using, and establish that the conditions of the theorem are satisfied.
- You do not have to answer every question to pass; indeed, it is better to answer fewer questions completely and correctly than to give partial answers to more questions. If you answer 4 questions completely and correctly you are guaranteed to pass.
- We use $\mathbb{R}$ to denote the real line, which is understood to be equipped with the usual topology and Lebesgue measure.
- $B(x, r)$ denotes the open ball of center x and radius r.
- For $1 \leq p \leq \infty$, $\|f\|_p$ denotes the L^p norm of f.

1. State the Open Mapping Theorem and the Closed Graph Theorem for Banach spaces. Derive the Open Mapping Theorem from the Closed Graph Theorem.

2. Let

$$\mathcal{H} = \left\{ u : \mathbb{R}^n \to \mathbb{R} : u \text{ continuous and } \forall x \in \mathbb{R}^n, \forall r > 0 :\ u(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} u(y)\, dy \right\}$$

   Show that if $\{u_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is uniformly bounded on compact subsets of $\mathbb{R}^n$, then there exists a subsequence that converges uniformly to a function $u \in \mathcal{H}$ on compact subsets of $\mathbb{R}^n$.

3. Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space, and suppose that λ is a **finitely** additive, nonnegative, real valued set function on $\mathcal{M}$ for which:

$$\forall\ \epsilon > 0, \exists\ \delta > 0,\ \forall E \in \mathcal{M} :\ \mu(E) < \delta \Rightarrow \lambda(E) < \epsilon.$$

   Prove that there exists $g \in L^1(\mu)$ such that for all $E \in \mathcal{M}$:

$$\lambda(E) = \int_E g\, d\mu.$$

4. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, and $\mathcal{F}$ is a family of non-negative integrable functions on X with the following properties:

   (a) If $\varphi,\ \psi \in \mathcal{F}$, then $\varphi + \psi \in \mathcal{F}$.

   (b) If $\varphi,\ \psi \in \mathcal{F}$, then $\max\{\varphi, \psi\} \in \mathcal{F}$.

   (c) If f is measurable, $f \geq 0$ and $\int f\, d\mu > 0$, then there is $\varphi \in \mathcal{F}$ such that $\varphi \leq f$ and $\int \varphi\, d\mu > 0$.

   Prove that if f is non-negative and measurable then

$$\int f\, d\mu = \sup\{\int \varphi\, d\mu : \varphi \in \mathcal{F}, \varphi \leq f\}$$

5. Let μ be a regular Borel measure on $\mathbb{R}^n$, and let m denote the Lebesgue measure on $\mathbb{R}^n$. Assume that there exists a constant $C > 1$ such that for all $x \in \mathbb{R}^n$ and $r > 0$:

$$C^{-1} r^n \leq \mu(B(x, r)) \leq C r^n.$$

   (a) Show that m and μ are mutually absolutely continuous.

   (b) Let f denote the Radon-Nikodym derivative of μ with respect to m (i.e. $d\mu = f\, dm$). Show that $f,\ \frac{1}{f} \in L^\infty(m)$.

6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$). Let $K : X \times X \to \mathbb{R}$ be an $\mathcal{M} \times \mathcal{M}$-measurable function satisfying

$$\int_X \int_X |K(x, y)|^2\, d\mu(x)\, d\mu(y) < \infty.$$

   Let $\{f_n\}_{n=1}^\infty \subset L^2(\mu)$ satisfy

$$\sup_n \|f_n\|_2 \leq 1.$$

   Assume also that there exists $f \in L^2(\mu)$ so that f_n converges weakly to f in $L^2(\mu)$, i.e. for all $g \in L^2(\mu)$:

$$\lim_{n\to\infty} \int_X f_n(y) g(y)\, d\mu(y) = \int_X f(y) g(y)\, d\mu(y).$$

   For $h \in L^2(\mu)$ define

$$Kh(x) = \int_X K(x, y) h(y)\, d\mu(y).$$

   Show that Kf_n converges strongly to Kf in $L^2(\mu)$ (i.e. $\|Kf_n - Kf\|_2 \to 0$ as $n \to \infty$).

7. (a) Let X be a topological space; let O be an open subset of X. Prove that the points of discontinuity of the characteristic function of O, χ_O, form a nowhere dense subset of X. Recall that $A \subset X$ is nowhere dense if $\operatorname{int} \overline{\mathrm{A}} = \emptyset$.

   (b) Let X be a complete metric space, and $\{U_i\}_{i=1}^{\infty}$ any countable collection of open sets. Show that there exists $x \in X$ such that χ_{U_i} is continuous at x for each i.

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$) and let $1 \leq p < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^p(\mu)$, and let f be an $\mathcal{M}$-measurable function such that f is finite μ-a.e. and $f_n \to f$ μ-a.e. Prove that $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f_n|^p \, d\mu < \epsilon \text{ whenever } \mu(E) < \delta \text{ and } E \in \mathcal{M}.$$