Real Analysis Preliminary Examination

Autumn, 2007

Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. Some problems have more than one part; the parts may not be weighted equally.

You may use any standard theorem from your real variables course, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it.

The reals $\mathbb{R}$ are assumed to be equipped with standard Lebesgue measure, and integrals with respect to this measure use the notation dx.

1. Let E and F be complex Banach spaces, and $T\colon E \to F$ be a continuous surjective linear map. Let $T^*\colon F^* \to E^*$ denote the adjoint map on dual spaces, defined by $(T^*f)(x) = f(Tx)$ for $x \in E$ and $f \in F^*$. Prove that there is a constant $c > 0$ such that $\|T^*f\| \geq c\|f\|$ for all $f \in F^*$.

2. (a) Let f be a nonnegative Lebesgue measurable function on $[0,\infty)$ such that $\int_0^\infty f(x)\,dx < \infty$. Show that there exists a positive, strictly increasing measurable function a on $[0,\infty)$ with $\lim_{x\to\infty} a(x) = \infty$ and such that

$$\int_0^\infty a(x)f(x)\,dx < \infty.$$

(b) Let $\{f_n\}$ be a sequence of nonnegative Lebesgue measurable functions on $[0,\infty)$ such that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = 0.$$

Show that there exists a positive, strictly increasing measurable function b on $[0,\infty)$ with $\lim_{x\to\infty} b(x) = \infty$ and such that

$$\lim_{n\to\infty} \int_0^\infty b(x)f_n(x)\,dx = 0.$$

3. Let M be a compact metric space with distance function d, and let Ω denote the set of all continuous functions $f\colon [0,1] \to M$ such that

$$d\big(f(s), f(t)\big) \leq |s-t| \quad \text{for all } s,t \in [0,1].$$

Define a metric ρ on Ω by the formula

$$\rho(f,g) = \sup_{0\leq t\leq 1} d\big(f(t), g(t)\big)$$

(you do not need to prove that ρ is a metric). Show that (Ω, ρ) is sequentially compact.

4. Let f be a strictly positive Borel measurable function on the reals $\mathbb{R}$, and let E be a Borel measurable subset of $\mathbb{R}$ with strictly positive Lebesgue measure. For every $t \in \mathbb{R}$ define

$$\phi(t) = \int_E f(t+x)\,dx.$$

(a) Prove that ϕ is Borel measurable from $\mathbb{R}$ to $[0, \infty]$.

(b) Suppose that $\phi \in L^1(\mathbb{R})$. Prove that E has finite Lebesgue measure and that $f \in L^1(\mathbb{R})$.

5. Let $1 \le p < \infty$, and let ℓ^p be the Banach space of pth power summable complex-valued sequences, so that $x = \{x_n\} \in \ell^p$ provide that

$$\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p} < \infty.$$

Let E be a closed subset of ℓ^p. Prove that E is compact in the norm topology on ℓ^p if and only if E satisfies the following two conditions:

(a) there is a constant C such that $\|x\|_p \le C$ for all $x \in E$,

(b) for every $\epsilon > 0$ there exists n_0 such that for all $x \in E$ we have that

$$\sum_{n=n_0}^{\infty} |x_n|^p < \epsilon\,.$$

6. Let X and Y be compact metric spaces, and let $C(X)$ and $C(Y)$ denote the Banach spaces of continuous, real-valued functions on X and Y, respectively, each equipped with the standard supremum norm. Suppose that $\phi\colon X \to Y$ is a continuous surjective map. Let

$$D = \{f \in C(X) : f(x) = f(x') \text{ whenever } \phi(x) = \phi(x')\}.$$

(a) Show that D is a closed subspace of $C(X)$, and that

$$D = \{g \circ \phi : g \in C(Y)\},$$

where $g \circ \phi(x) = g(\phi(x))$ for all $x \in X$.

(b) Let ν be a finite positive Borel measure on Y. Prove that there is a finite positive Borel measure μ on X such that $\mu\big(\phi^{-1}(F)\big) = \nu(F)$ for all Borel subsets F of Y.

7. Let X be a compact metric space, and μ be a finite positive Borel measure on X. Suppose that $\mu(\{x\}) = 0$ for every $x \in X$. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that, if E is any Borel subset of X having diameter less than δ, then $\mu(E) < \epsilon$.

8. Let T be the triangle $\{(x, y) \in \mathbb{R}^2 : 0 \le |x| \le y \le 1\}$, and μ be the restriction of planar Lebesgue measure to T. Suppose that $f \in L^2(T, \mu)$. Prove that

$$\liminf_{y \to 0^+} \int_{-y}^{y} |f(x, y)|\,dx = 0.$$