REAL ANALYSIS PRELIMINARY EXAM

September 9, 2008

Do as many of the eight problems as you can. If any questions or instructions are not clear, ask the proctor. Four completely correct solutions will be regarded as a clear pass. Keep in mind that complete solutions are better than partial solutions. It also helps in partial solutions to clearly indicate where the gaps are. Always carefully justify your answers.

You may use any standard theorem from your real analysis course, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it. The notation $\mathbb{R}$ stands for the real numbers, equipped with standard Lebesgue measure and the integral with respect to this measure uses the notation dx. In particular, $L^2(\mathbb{R}, dx)$ denotes the L^2-space on $\mathbb{R}$ with respect to Lebesgue measure.

1. Let $g \in L^2(\mathbb{R}, dx)$, and set $f(x) = \int_0^x g(t)\,dt,\ x \in \mathbb{R}$.
   (a) Show by example that f need not be differentiable at 0.
   (b) Must f have any points of differentiability? Explain.
   (c) Let $\varphi(x) = f(x)^2$. Show that φ is differentiable at 0 and find $\varphi'(0)$.

2. Evaluate
$$\lim_{n\to\infty} n^{3/2} \int_0^1 \frac{x^2}{(1+x^2)^n}\,dx.$$
   Be sure to provide justification for your argument.

3. Let X be a compact Hausdorff space. Suppose that $\{f_n, n \geq 1\}$ is a sequence of continuous real-valued functions on X such that $f_{n+1}(x) \leq f_n(x)$ for every $n \geq 1$ and $x \in X$. Suppose also that $f(x) = \lim_{n\to\infty} f_n(x)$ is finite for every $x \in X$. Show that if the limiting function f is continuous on X, then f_n converges to f uniformly on X.

4. Suppose that H is a separable real Hilbert space with an orthonormal basis $\{e_k, k \geq 1\}$ and with inner product denoted by $\langle \cdot, \cdot \rangle$. Let $\{y_k, k \geq 1\} \subset H$. Prove that the following two statements are equivalent.
   (i) $\lim_{k\to\infty} \langle x, y_k \rangle = 0$ for every $x \in H$;
   (ii) $\sup_{k\geq 1} \|y_k\| < \infty$ and $\lim_{k\to\infty} \langle e_n, y_k \rangle = 0$ for every $n \geq 1$.

(over)

5. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $1 < p < \infty$ and denote by $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$ the L^p-norm of a function $f \in L^p(X, \mu)$.

   Let f_n, $f \in L^p(X, \mu)$. Suppose that $\sup_{n \geq 1} \|f_n\|_p < \infty$ and that $f_n \to f$ a.e. Show that $f_n \to f$ weakly in L^p; that is,

$$\lim_{n \to \infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu \qquad \text{for every } g \in L^q(X, \mu),$$

   where $\frac{1}{p} + \frac{1}{q} = 1$.

6. Let (X, d) be a metric space. A map $f : X \to X$ is said to be an *isometry* of (X, d) if $d\left(f(x), f(y)\right) = d(x, y)$ for all x, $y \in X$. Show that if (X, d) is a compact metric space and $f : X \to X$ is an isometry, then f is surjective.

7. Let E be the set of real numbers in $[0, 1]$ whose decimal expansion contains an infinite number of 7's. Show that E is Lebesgue measurable and determine, with proof, the Lebesgue measure of E.

8. Let f be a positive, continuously differentiable function on $(0, \infty)$ satisfying $f'(x) > 0$ for all $x \in (0, \infty)$. Suppose that for some constant $C > 0$,

$$f(x) \leq C x^2 \qquad \text{for} \quad x \geq 1.$$

   Show that

$$\int_0^\infty \frac{1}{f'(x)} \, dx = \infty.$$

   (*Hint*: Establish and use the fact that if $\displaystyle\int_0^\infty \frac{1}{f'(x)} \, dx < \infty$, then $\displaystyle\lim_{a \to \infty} \int_a^\infty \frac{1}{f'(x)} \, dx = 0$.)