2009 REAL ANALYSIS PRELIM

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation:

- All functions in this exam are real-valued.
- Lebesgue measure on the real line \( \mathbb{R} \) is denoted by \( \lambda \). Integrals with respect to \( \lambda \) are written as \( \int f(x) \, dx \).
- If \( f \) is a function on a set \( X \), we set \( \| f \|_{\sup} = \sup_{x \in X} |f(x)| \).
- If \( X \) is a compact Hausdorff space, \( C(X) \) is the space of continuous functions on \( X \), equipped with the norm \( \| \cdot \|_{\sup} \).

1. Suppose that \( A \) is a Borel subset of \([0, 1] \times [0, 1]\), and let

\[
A_x = \{ y \in [0, 1] : (x, y) \in A \}, \quad A^y = \{ x \in [0, 1] : (x, y) \in A \},
\]

\[
B = \{ x \in [0, 1] : \lambda(A_x) = 1/3 \}.
\]

Suppose that \( \lambda(B) = 1/5 \). Prove that there exists \( y \in [0, 1] \) such that \( \lambda(A^y) \leq 13/15 \).

2. In this problem, \( L^p = L^p(X, \mu) \) where \( (X, \mu) \) is a measure space. Suppose \( 1 < p < 2 \) and \( (1/p) + (1/q) = 1 \).

a. Show that \( L^p \cap L^q \) is complete (and hence is a Banach space) with respect to the norm \( \| f \|_{p,q} = \| f \|_p + \| f \|_q \).

b. Show that if \( f \in L^p \cap L^q \), then \( f \in L^2 \) and \( \| f \|_2 \leq \sqrt{\| f \|_p \| f \|_q} \).

c. Show that \( L^p \cap L^q \) is dense in \( L^2 \) with respect to the \( L^2 \) norm.

d. Since \( \sqrt{ab} \leq \frac{1}{2}(a + b) \) for all \( a, b \geq 0 \), part (b) implies that \( \| f \|_2 \leq \frac{1}{2} \| f \|_{p,q} \). Show that if there is a constant \( c > 0 \) such that \( \| f \|_2 \geq c \| f \|_{p,q} \) for all \( f \in L^p \cap L^q \), then \( L^p \cap L^q = L^2 \). Give an example of a space \((X, \mu)\) where this is the case.

3. The Fourier cosine transform of a function \( f \in L^1(\mathbb{R}, \lambda) \) is the function \( \hat{f} \) on \( \mathbb{R} \) defined by

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} \cos(\omega t) f(t) \, dt.
\]

Prove the “Riemann-Lebesgue lemma”: if \( f \in L^1(\mathbb{R}, \lambda) \) then \( \hat{f} \) is continuous and vanishes at infinity. You can assume without proof that the functions of class \( C^1 \) that vanish outside a bounded interval are dense in \( L^1(\mathbb{R}, \lambda) \).
4. Let $K$ be a continuous function on $[0,1] \times [0,1]$. For $f \in L^2([0,1], \lambda)$ and $x \in [0,1]$, define $Tf(x) = \int_0^1 K(x, y) f(y) dy$.
   a. Show that $\|Tf\|_{\sup} \leq \|K\|_{\sup} \|f\|_2$ for all $x \in [0,1]$, and that $Tf$ is continuous.
   b. Show that if \{${f_n}$\} is a bounded sequence in $L^2([0,1], \lambda)$, the sequence \{${Tf_n}$\} contains a uniformly convergent subsequence.
   c. Assume that $T$ is one-to-one. Show that $T$ does not map $L^2([0,1], \lambda)$ onto $C([0,1])$.

5. A sequence \{${f_n}$\} in $C([0,1])$ is said to converge weakly to $f$ in $C([0,1])$ if $\phi(f_n) \to \phi(f)$ for every bounded linear functional $\phi$ on $C([0,1])$.
   a. Show that $f_n \to f$ weakly in $C([0,1])$ if and only if $f_n \to f$ pointwise and there is a constant $C$ such that $\|f_n\|_{\sup} \leq C$ for all $n \geq 1$.
   b. Show that if $f_n \to f$ weakly in $C([0,1])$, then $f_n \to f$ in the $L^p$ norm with respect to Lebesgue measure for all $p \in [1,\infty)$.

6. Let $\{q_n\}_{n=1}^{\infty}$ be the set of rational numbers in $[0,1]$, ordered in some way. Prove that the series
   \[
   \sum_{n=1}^{\infty} (-1)^n n^{-3/2} |x - q_n|^{-q_n + (1/2)}
   \]
   converges to a finite limit for $\lambda$-almost all $x \in [0,1]$.

7. Let $\mathcal{K}$ be the family of all nonempty compact subsets of $\mathbb{R}$. For $A, B \in \mathcal{K}$, define
   \[
   d(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y| + \inf_{y \in B} \sup_{x \in A} |x - y|.
   \]
   a. Prove that $(\mathcal{K}, d)$ is a metric space.
   b. Prove that $(\mathcal{K}, d)$ is separable.

8. Let $X$ be a compact Hausdorff space. A function $f : X \to \mathbb{R}$ is called upper semi-continuous if for every $x \in X$ and every $\epsilon > 0$ there is a neighborhood $U$ of $x$ such that $f(y) < f(x) + \epsilon$ for all $y \in U$.
   a. Show that $f$ is upper semi-continuous if and only if \{${x : f(x) < a}$\} is open in $X$ for every $a \in \mathbb{R}$.
   b. Show that if $f$ is upper semi-continuous, there exists $K \in \mathbb{R}$ such that $f(x) < K$ for all $x \in X$.
   c. Show that if $f$ is upper semi-continuous, then
   \[
   f(x) = \inf \{g(x) : g \in C(X) \text{ and } g(y) > f(y) \text{ for all } y \in X\}.
   \]