2009 REAL ANALYSIS PRELIM

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation:

- All functions in this exam are real-valued.
- Lebesgue measure on the real line $\mathbb{R}$ is denoted by λ. Integrals with respect to λ are written as $\int f(x)\,dx$.
- If f is a function on a set X, we set $\|f\|_{\sup} = \sup_{x\in X} |f(x)|$.
- If X is a compact Hausdorff space, $C(X)$ is the space of continuous functions on X, equipped with the norm $\|\cdot\|_{\sup}$.

1. Suppose that A is a Borel subset of $[0,1]\times[0,1]$, and let

$$A_x = \big\{y\in[0,1] : (x,y)\in A\big\},\quad A^y = \big\{x\in[0,1] : (x,y)\in A\big\},$$
$$B = \big\{x\in[0,1] : \lambda(A_x) = 1/3\big\}.$$

   Suppose that $\lambda(B) = 1/5$. Prove that there exists $y\in[0,1]$ such that $\lambda(A^y)\le 13/15$.

2. In this problem, $L^p = L^p(X,\mu)$ where (X,μ) is a measure space. Suppose $1<p<2$ and $(1/p)+(1/q)=1$.

   a. Show that $L^p\cap L^q$ is complete (and hence is a Banach space) with respect to the norm $\|f\|_{p,q} = \|f\|_p + \|f\|_q$.
   b. Show that if $f\in L^p\cap L^q$, then $f\in L^2$ and $\|f\|_2 \le \sqrt{\|f\|_p\|f\|_q}$.
   c. Show that $L^p\cap L^q$ is dense in L^2 with respect to the L^2 norm.
   d. Since $\sqrt{ab}\le \frac{1}{2}(a+b)$ for all $a,b\ge 0$, part (b) implies that $\|f\|_2 \le \frac{1}{2}\|f\|_{p,q}$. Show that if there is a constant $c>0$ such that $\|f\|_2 \ge c\|f\|_{p,q}$ for all $f\in L^p\cap L^q$, then $L^p\cap L^q = L^2$. Give an example of a space (X,μ) where this is the case.

3. The Fourier cosine transform of a function $f\in L^1(\mathbb{R},\lambda)$ is the function $\widehat{f}$ on $\mathbb{R}$ defined by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} \cos(\omega t) f(t)\,dt.$$

   Prove the "Riemann-Lebesgue lemma": if $f\in L^1(\mathbb{R},\lambda)$ then $\widehat{f}$ is continuous and vanishes at infinity. You can assume without proof that the functions of class C^1 that vanish outside a bounded interval are dense in $L^1(\mathbb{R},\lambda)$.

4. Let K be a continuous function on $[0,1] \times [0,1]$. For $f \in L^2([0,1], \lambda)$ and $x \in [0,1]$, define $Tf(x) = \int_0^1 K(x,y) f(y)\, dy$.
   a. Show that $\|Tf\|_{\sup} \leq \|K\|_{\sup} \|f\|_2$ for all $x \in [0,1]$, and that Tf is continuous.
   b. Show that if $\{f_n\}$ is a bounded sequence in $L^2([0,1], \lambda)$, the sequence $\{Tf_n\}$ contains a uniformly convergent subsequence.
   c. Assume that T is one-to-one. Show that T does not map $L^2([0,1], \lambda)$ *onto* $C([0,1])$.

5. A sequence $\{f_n\}$ in $C([0,1])$ is said to *converge weakly* to f in $C([0,1])$ if $\phi(f_n) \to \phi(f)$ for every bounded linear functional ϕ on $C([0,1])$.
   a. Show that $f_n \to f$ weakly in $C([0,1])$ if and only if $f_n \to f$ pointwise and there is a constant C such that $\|f_n\|_{\sup} \leq C$ for all $n \geq 1$.
   b. Show that if $f_n \to f$ weakly in $C([0,1])$, then $f_n \to f$ in the L^p norm with respect to Lebesgue measure for all $p \in [1, \infty)$.

6. Let $\{q_n\}_1^\infty$ be the set of rational numbers in $[0,1]$, ordered in some way. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n n^{-3/2} |x - q_n|^{-q_n + (1/2)}$$

converges to a finite limit for λ-almost all $x \in [0,1]$.

7. Let $\mathcal{K}$ be the family of all nonempty compact subsets of $\mathbb{R}$. For $A, B \in \mathcal{K}$, define

$$d(A,B) = \sup_{x \in A} \inf_{y \in B} |x - y| + \sup_{y \in B} \inf_{x \in A} |x - y|.$$

   a. Prove that $(\mathcal{K}, d)$ is a metric space.
   b. Prove that $(\mathcal{K}, d)$ is separable.

8. Let X be a compact Hausdorff space. A function $f : X \to \mathbb{R}$ is called *upper semi-continuous* if for every $x \in X$ and every $\epsilon > 0$ there is a neighborhood U of x such that $f(y) < f(x) + \epsilon$ for all $y \in U$.
   a. Show that f is upper semi-continuous if and only if $\{x : f(x) < a\}$ is open in X for every $a \in \mathbb{R}$.
   b. Show that if f is upper semi-continuous, there exists $K \in \mathbb{R}$ such that $f(x) < K$ for all $x \in X$.
   c. Show that if f is upper semi-continuous, then

$$f(x) = \inf\big\{g(x) : g \in C(X) \text{ and } g(y) > f(y) \text{ for all } y \in X\big\}.$$