REAL ANALYSIS PRELIM 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

A few definitions / notations we use below.

- A probability measure is a positive measure such that the total mass is one.
- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- Sequences will be denoted by the notation $\langle a_n \rangle$. Be careful: note that $\langle x, y \rangle$ refers to an inner product.
- $C[0,1]$ - the set of real-valued continuous functions on the interval $[0,1]$.

(1) Prove that for all $t > 0$, we have
$$\int_0^\infty \frac{e^{-x} - e^{-xt}}{x} dx = \ln t.$$
Justify all integral manipulations.

(2) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, with μ being a probability measure. Suppose φ is a continuous real-valued function on $\mathbb{R}$ such that
$$\varphi\left(\int_\Omega f(x) d\mu(x)\right) \leq \int_\Omega \varphi(f(x)) d\mu(x)$$
for every real bounded measurable f.

Suppose that there is a measurable subset A such that $\mu(A) = 1/2$. Prove that φ is then convex.

Note that this is a partial converse to Jensen's inequality for convex functions.

(3) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\langle [f_n] \rangle$ be a sequence in $\mathbf{L}^p(\mu)$, $1 < p < \infty$, where $[h]$ refers to the function class of h. Suppose f_n converges to f almost everywhere and $\left\langle \int f_n g d\mu \right\rangle$ is bounded for all $[g] \in \mathbf{L}^q(\mu)$, where $1/p + 1/q = 1$. Prove that $[f] \in \mathbf{L}^p(\mu)$.

(4) Suppose $1 < p < \infty$. Let $f \in \mathbf{L}^p(0, \infty)$, relative to the Lebesgue measure, and set
$$F(x) = \frac{1}{x} \int_0^x f(t) dt, \quad 0 < x < \infty.$$
Prove Hardy's inequality:
$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p .$$

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Hint: First explain why it suffices to consider $f \geq 0$. Then, start with the case when f is continuous, nonnegative, and compactly supported in the open interval $(0, \infty)$, and apply integration by parts to the integral $\int_0^\infty F^p(x)dx$.

(5) Suppose μ is a finite signed Borel measure on $[0, 1]$ such that

$$\int_0^1 e^{-nx} d\mu(x) = 0, \quad \text{for all } n \in \mathbb{N}.$$

Prove that $\mu = 0$.

(6) Let $(\Omega, \mathcal{F})$ be a measurable space on which two probability measures μ and ν exist. Define the Total Variation metric between the two measures as

$$\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|.$$

Suppose λ is any positive measure on $\mathcal{F}$ such that both μ and ν are dominated by λ. Prove that

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \int_\Omega \left| \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right| d\lambda.$$

Here $d\mu/d\lambda$ and $d\nu/d\lambda$ refer to Radon-Nikodým derivatives.

(7) Suppose $\mathbf{X}$ is a Hausdorff space, $(\mathbf{Y}, \mathbf{d})$ is a metric space, and $\langle f_n \rangle$ is an equicontinuous sequence of functions from $\mathbf{X}$ to $\mathbf{Y}$. That is, for all $p \in \mathbf{X}$ and for all $\epsilon > 0$, there exists an open neighborhood U of p such that

$$\mathbf{d}\left(f_n(x), f_n(p)\right) < \epsilon \quad \text{for all } n, \text{ whenever} \quad x \in U.$$

Prove that

$$C = \{x \in \mathbf{X} : \ \langle f_n(x) \rangle \text{ is a Cauchy sequence in } \mathbf{Y}\}$$

is closed in $\mathbf{X}$.

(8) Let $C^1[0, 1]$ be the set $\{f \in C[0, 1] : \text{ there exists } f' \in C[0, 1]\}$, where we take the right derivative of f at zero and the left derivative at one in defining f'. Equip $C^1[0, 1]$ with the inner product

$$\langle f, g \rangle_1 = \int_0^1 f(t)g(t)dt + \int_0^1 f'(t)g'(t)dt.$$

This inner product, in a natural way, induces a norm and a metric on $C^1[0, 1]$.

Prove that any Cauchy sequence from $C^1[0, 1]$ in the above metric converges ($\mathbf{L}^2$-sense) to a continuous function. In other words, the completion of $C^1[0, 1]$ in this metric can be taken to be a subset of $C[0, 1]$.