REAL ANALYSIS PRELIM 2010

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

A few definitions / notations we use below.

- A probability measure is a positive measure such that the total mass is one.
- \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \).
- Sequences will be denoted by the notation \( \langle a_n \rangle \). Be careful: note that \( \langle x, y \rangle \) refers to an inner product.
- \( C[0, 1] \) - the set of real-valued continuous functions on the interval \([0, 1]\).

(1) Prove that for all \( t > 0 \), we have
\[
\int_0^\infty \frac{e^{-x} - e^{-xt}}{x} dx = \ln t.
\]
Justify all integral manipulations.

(2) Let \((\Omega, \mathcal{F}, \mu)\) be a measure space, with \( \mu \) being a probability measure. Suppose \( \varphi \) is a continuous real-valued function on \( \mathbb{R} \) such that
\[
\varphi \left( \int_{\Omega} f(x) d\mu(x) \right) \leq \int_{\Omega} \varphi(f(x)) d\mu(x)
\]
for every real bounded measurable \( f \).

Suppose that there is a measurable subset \( A \) such that \( \mu(A) = 1/2 \).
Prove that \( \varphi \) is then convex.

Note that this is a partial converse to Jensen’s inequality for convex functions.

(3) Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. Let \( \langle [f_n] \rangle \) be a sequence in \( L^p(\mu) \), \( 1 < p < \infty \), where \([h]\) refers to the function class of \( h \). Suppose \( f_n \) converges to \( f \) almost everywhere and \( \langle f, f_n g d\mu \rangle \) is bounded for all \( [g] \in L^q(\mu) \), where \( 1/p + 1/q = 1 \).
Prove that \( [f] \in L^p(\mu) \).

(4) Suppose \( 1 < p < \infty \). Let \( f \in L^p(0, \infty) \), relative to the Lebesgue measure, and set
\[
F(x) = \frac{1}{x} \int_0^x f(t) dt, \quad 0 < x < \infty.
\]
Prove Hardy’s inequality:
\[
\|F\|_p \leq \frac{p}{p-1} \|f\|_p.
\]

Date: September 10, 2010.
Hint: First explain why it suffices to consider \( f \geq 0 \). Then, start with the case when \( f \) is continuous, nonnegative, and compactly supported in the open interval \((0, \infty)\), and apply integration by parts to the integral \( \int_0^\infty F^p(x)dx \).

(5) Suppose \( \mu \) is a finite signed Borel measure on \([0, 1]\) such that
\[
\int_0^1 e^{-nx}d\mu(x) = 0, \quad \text{for all } n \in \mathbb{N}.
\]
Prove that \( \mu = 0. \)

(6) Let \((\Omega, \mathcal{F})\) be a measurable space on which two probability measures \( \mu \) and \( \nu \) exist. Define the Total Variation metric between the two measures as
\[
\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|.
\]
Suppose \( \lambda \) is any positive measure on \( \mathcal{F} \) such that both \( \mu \) and \( \nu \) are dominated by \( \lambda \). Prove that
\[
\|\mu - \nu\|_{TV} = \frac{1}{2} \int_\Omega \left| \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right| d\lambda.
\]
Here \( d\mu/d\lambda \) and \( d\nu/d\lambda \) refer to Radon-Nikodym derivatives.

(7) Suppose \( X \) is a Hausdorff space, \((Y, d)\) is a metric space, and \( \langle f_n \rangle \) is an equicontinuous sequence of functions from \( X \) to \( Y \). That is, for all \( p \in X \) and for all \( \epsilon > 0 \), there exists an open neighborhood \( U \) of \( p \) such that
\[
d(f_n(x), f_n(p)) < \epsilon \quad \text{for all } n, \text{ whenever } x \in U.
\]
Prove that
\[
C = \{ x \in X : \langle f_n(x) \rangle \text{ is a Cauchy sequence in } Y \}
\]
is closed in \( X \).

(8) Let \( C^1[0, 1] \) be the set \( \{ f \in C[0, 1] : \text{there exists } f' \in C[0, 1] \} \), where we take the right derivative of \( f \) at zero and the left derivative at one in defining \( f' \). Equip \( C^1[0, 1] \) with the inner product
\[
\langle f, g \rangle_1 = \int_0^1 f(t)g(t)dt + \int_0^1 f'(t)g'(t)dt.
\]
This inner product, in a natural way, induces a norm and a metric on \( C^1[0, 1] \).

Prove that any Cauchy sequence from \( C^1[0, 1] \) in the above metric converges (\( L^2 \)-sense) to a continuous function. In other words, the completion of \( C^1[0, 1] \) in this metric can be taken to be a subset of \( C[0, 1] \).