REAL ANALYSIS PRELIM 2012

Do as many of the eight problems as you can. Ask the proctor if any question or instruction is not clear. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. In case of a partial proof, point out the gap and, if possible, indicate what would be required to fill it in. You may use any standard theorem from your real analysis course, either identifying it by name or stating it in full.

A few definitions / notations we use below.

- $\mathbb{R}$ stands for the real line.
- $\mathbf{L}^p(\mathbb{R}, dx)$ denotes the $\mathbf{L}^p$ space of functions on the real line equipped with the Lebesgue measure.

Problem 1. Let A and B be two topological spaces such that there exists a homeomorphism f from $[0,1]$ to $A \times B$, equipped with the product topology. Prove that one of A or B must be a singleton.

Hint: Try to argue by contradiction. Use the fact that a homeomorphism preserves path-connectedness.

Problem 2. A metric space is said to be separable if it has a countable dense subset. Suppose μ is a finite Borel measure on a metric space (X, d). We say μ is tight if, for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq X$ such that $\mu(K_\epsilon) > \mu(X)(1 - \epsilon)$. Show that, if X is complete and separable, every finite measure μ is tight.

Problem 3. Let I be an open interval of $\mathbb{R}$. Let f be a convex function that is finite and differentiable on I. Let $f_1, f_2, \ldots$ be a sequence of convex functions, each finite and differentiable on I, such that $\lim_{i\to\infty} f_i(x) = f(x)$ for each $x \in I$. Then prove that

$$\lim_{i\to\infty} f_i'(x) = f'(x), \quad \text{for all } x \in I.$$

Hint: Try to argue by contradiction. The convexity assumption is crucial.

Problem 4. Let m denote the Lebesgue measure on $\mathbb{R}$ and let $\mathcal{F}$ be the σ-field of Lebesgue measurable subsets of $\mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, i.e., there is a positive constant Λ such that $|f(x) - f(y)| \le \Lambda |x - y|$, for every $x, y \in \mathbb{R}$. Show that if $A \in \mathcal{F}$, then $f(A) \in \mathcal{F}$, and $m(f(A)) \le \Lambda m(A)$.

Problem 5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For some $1 < p < \infty$ consider the $\mathbf{L}^p$ space corresponding to the measure space $(\Omega, \mathcal{F}, \mu)$. Let f, g be two functions in $\mathbf{L}^p$. For $u \in \mathbb{R}$, define the function $F(u) = \int_\Omega |f + ug|^p \, d\mu$. Show that this function is differentiable at zero and

$$F'(0) = p \int_\Omega |f|^{p-2} fg \, d\mu.$$

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Hint: Reduce the problem to justifying taking a limit under the integral.

Problem 6. Let $(f_n,\ n \geq 1)$ be a sequence in $\mathbf{L}^p(\mathbb{R}, dx)$, for some $p \geq 1$. Let f be another element in $\mathbf{L}^p(\mathbb{R}, dx)$. Show that the following two assertions are equivalent.

(i) $\sup_{n\geq 1} \|f_n\|_p < \infty$ and $\lim_{n\to\infty} \int_0^x f_n(t)dt = \int_0^x f(t)dt$ for every $x \in \mathbb{R}$.
(ii) $(f_n,\ n \geq 1)$ converges weakly to f in $\mathbf{L}^p(\mathbb{R}, dx)$.

Problem 7. Prove that any finite dimensional subspace of a normed linear space is closed.

Problem 8. A subset C of a Hilbert space $(H, \|\cdot\|)$ is said to be convex if for all $x, y \in C$ and all $0 \leq t \leq 1$, the linear combination $tx + (1-t)y \in C$. Show that, given a nonempty, closed, convex subset $C \subseteq H$, any element $x \in H$ has a unique element $y \in C$ such that

$$\|x - y\| \leq \|x - z\| \quad \text{for all } z \in C.$$

Hint: Consider a minimizing sequence and argue that it is Cauchy.