Real Analysis Preliminary Examination

Autumn, 2013

Instructions: The exam is four hours long. There are eight problems, each weighted equally. Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

You may use any standard theorem from your real variables course, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it.

The reals $\mathbb{R}$ are assumed to be equipped with standard Lebesgue measure denoted by m, and integrals with respect to this measure use the notation dx. All functions are assumed to be real-valued unless otherwise stated.

1. Suppose that $f_n\colon [0,1] \to \mathbb{R}$, $(n \geq 1)$, are Lebesgue measurable, and that

$$(*) \qquad \int_0^1 |f_n(x)|^3 \, dx \leq 1 \quad \text{for all } n \geq 1.$$

(a) Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every Lebesgue measurable subset $E \subset [0,1]$ with $m(E) < \delta$ we have that

$$\int_E |f_n(x)| \, dx < \epsilon \quad \text{for all } n \geq 1\,.$$

(b) Give an example, with proof, that the conclusion of (a) is false if (*) is replaced by

$$\int_0^1 |f_n(x)| \, dx \leq 1 \quad \text{for all } n \geq 1.$$

2. A topological space is called *separable* if it contains a dense subset that is finite or countably infinite.

(a) Prove that every subspace of a separable metric space is also separable.

(b) Provide an example of a separable topological space which has a subspace that is not separable.

3. Let S be a subset of $\mathbb{R}$ with strictly positive Lebesgue measure, and let $\mathbb{Q}$ denote the set of rational numbers in $\mathbb{R}$. Prove that almost every (with respect to Lebesgue measure) real number can be written as the sum of an element of S and an element of $\mathbb{Q}$.

4. Let H be a separable, infinite-dimensional Hilbert space over the reals. For a bounded linear operator $T\colon H \to H$, recall that its norm is defined by $\|T\| = \sup\{\|Tx\| : x \in H, \|x\| = 1\}$.

(a) If T is a bounded linear operator on H such that $\|I - T\| < 1$, where I is the identity operator on H, prove that T is invertible with a bounded inverse.

(b) Suppose that $\{e_n : n \geq 1\}$ is a complete orthonormal set in H. Suppose also that $\{f_n : n \geq 1\}$ is an orthonormal set in H such that

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1\,.$$

Prove that $\{f_n : n \geq 1\}$ is a *complete* orthonormal set in H.

5. Let $f \in L^1(\mathbb{R})$. Prove that the following limits exist, and determine their values in terms of $\|f\|_1$.

(a)
$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) + f(x)|\, dx.$$

(b)
$$\lim_{t\to \infty} \int_{\mathbb{R}} |f(x+t) + f(x)|\, dx.$$

6. Let X be a compact Hausdorff space, and let $C(X)$ denote the space of continuous, complex-valued functions on X. Denote as usual the sup norm on $C(X)$ by $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$.

Suppose there is another norm $\|\cdot\|$ on $C(X)$ under which is it also a Banach space. Assume moreover that, for each $x \in X$ the linear functional λ_x, defined by $\lambda_x(f) = f(x)$, is bounded on this Banach space. Show that there exist strictly positive constants A and B such that, for every $f \in C(X)$, we have that

$$A\,\|f\|_\infty \leq \|f\| \leq B\,\|f\|_\infty\,.$$

7. Let $\{f_n : n \geq 1\}$ be a sequence of continuously differentiable functions on $[0, 1]$, and assume that

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \quad \text{for all } 0 < x \leq 1 \text{ and all } n \geq 1,$$

and that

$$\int_0^1 f_n(x)\, dx = 0 \quad \text{for all } n \geq 1\,.$$

Prove that this sequence has a subsequence that converges uniformly on $[0, 1]$.

8. Let f be a continuous real-valued function on $[0, \infty)$ with $f(0) = 0$. Suppose that for each $y \in [0, 1]$ we have that $f(ny) \to 0$ as $n \to \infty$ through the integers. Prove that $f(x) \to 0$ as $x \to \infty$ through the reals.