Real Analysis Prelim – 2015

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation:

- $\mathbb{R}$ is the real line; $[a,b] \subset \mathbb{R}$ is the closed interval from a to b.
- m denotes Lebesgue measure. All integrals on $\mathbb{R}$ or on subsets of $\mathbb{R}$ assume that Lebesgue measure is used.
- $L^p([a,b])$ is the space of p-integrable functions on $[a,b]$ with the usual norm.

1. Suppose that $1 < p < \infty$, and $\{f_j\}_{j=1}^\infty$ is a sequence in $L^1\big([0,1]\big)$ such that

$$\sup_j \int_{[0,1]} |f_j|^p \, dm < \infty \,.$$

(a.) If $f \in L^1\big([0,1]\big)$ is such that $\lim_{j\to\infty} \|f_j - f\|_{L^1} = 0$, show that

$$\int_{[0,1]} |f|^p \, dm < \infty \,.$$

(b.) Is it necessarily true that $\lim_{j\to\infty} \|f_j - f\|_{L^p} = 0$? Either prove that it's true, or give a counterexample.

2. For $1 \le p \le \infty$, and $f \in L^p\big([0,1]\big)$, define

$$\big(Tf\big)(x) = \int_0^x f(s)\, ds \,, \qquad 0 \le x \le 1 \,.$$

(a.) Show that, if $1 < p \le \infty$, T is a compact linear mapping of $L^p\big([0,1]\big)$ into $C\big([0,1]\big)$, where $C\big([0,1]\big)$ is the space of continuous functions on $[0,1]$ with the uniform norm.

(b.) Is this true for $p = 1$? Either prove that it is true, or give a counterexample.

3. Suppose that $\{f_j\}_{j=1}^\infty$ is a sequence in $L^2(\mathbb{R})$ such that f_j converges weakly to $f \in L^2(\mathbb{R})$.

(a.) Give an example of such a sequence for which $\lim\limits_{j\to\infty} \|f_j - f\|_{L^2} \neq 0$.

(b.) Show that if $\lim\limits_{j\to\infty} \|f_j\|_{L^2} = \|f\|_{L^2}$ then $\lim\limits_{j\to\infty} \|f_j - f\|_{L^2} = 0$.

4. Suppose that f is a real-valued Borel measurable function on the interval $[0,1]$. Show that there exists a sequence of polynomial functions $\{P_k(x)\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} P_k(x) = f(x) \text{ pointwise almost everywhere on } [0,1] \,.$$

5. Let $\ell^2(\mathbb{N})$ denote the Hilbert space of square summable, complex valued sequences $\mathbf{a} = \{a_j\}_{j=1}^{\infty}$, with norm

$$\|\mathbf{a}\|_{\ell^2(\mathbb{N})} = \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{\frac{1}{2}}.$$

If K is a compact subset of $\ell^2(\mathbb{N})$, show that there exists a sequence of real numbers $\{m_j\}_{j=1}^{\infty}$ satisfying the following conditions

$$0 < m_1 \le m_2 \le m_3 \le \cdots, \qquad \lim_{j\to\infty} m_j = \infty,$$

such that for every $\mathbf{a} \in K$ the following holds

$$\sum_{j=1}^{\infty} m_j |a_j|^2 \le 1.$$

6. Suppose that $E \subset \mathbb{R}$ is a measurable subset of strictly positive measure. Show that there exists $\delta_0 > 0$ such that, for all $0 \le \delta < \delta_0$,

$$m\big(E \cap (E+\delta)\big) > 0, \qquad \text{where} \quad E + \delta = \{x + \delta : x \in E\}.$$

7. Recall that weak-L^p is the space of measurable functions f such that $[f]_p < \infty$, where

$$[f]_p = \left(\sup_{\alpha > 0} \alpha^p \, m\big(\{x : |f(x)| > \alpha\}\big) \right)^{\frac{1}{p}}.$$

(a.) Show that if f belongs to both weak-L^1 and weak-L^2, then $f \in L^p$ for $1 < p < 2$.

(b.) Show that for each $r > 0$,

$$\int |f|^p \;\le\; \frac{p}{p-1}\,[f]_1\, r^{\,p-1} \;+\; \frac{p}{2-p}\,[f]_2^2\, r^{\,p-2}$$

What value of r makes the right side minimal?

8. Define functions $u(x), v(x) \in L^1_{\text{loc}}(\mathbb{R})$ as follows:

$$u(x) = \begin{cases} \ln(x), & x > 0 \\ 0, & x \le 0 \end{cases} \qquad\qquad v(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$$

and corresponding distributions $u, v \in \mathcal{D}'(\mathbb{R})$ by

$$\langle u, \phi \rangle = \int_{\mathbb{R}} u(x)\,\phi(x)\,dx, \qquad \langle v, \phi \rangle = \int_{\mathbb{R}} v(x)\,\phi(x)\,dx, \qquad \phi \in C_c^{\infty}(\mathbb{R}).$$

(a.) Show that $x\partial u = v$ (in the sense of distributions in $\mathcal{D}'(\mathbb{R})$).

(b.) For $f \in \mathcal{D}'(\mathbb{R})$ and $r > 0$, let $f_r \in \mathcal{D}'(\mathbb{R})$ denote the dilation of f by r. Find the distribution $(\partial u)_r - r^{-1}\partial u$.

Note: For $f \in L^1_{\text{loc}}(\mathbb{R})$ dilation is defined by $f_r(x) = f(rx)$. You may use that $\partial f_r = r(\partial f)_r$ holds for $f \in \mathcal{D}'(\mathbb{R})$.