REAL ANALYSIS PRELIMINARY EXAM

September, 2016

Do as many of the eight problems as you can. If any questions or instructions are not clear, ask the proctor. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

You may use any standard theorem from your real analysis course, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it. The notation $\mathbb{R}$ stands for the real numbers, equipped with standard Lebesgue measure and the integral with respect to this measure uses the notation dx. In particular, $L^2(\mathbb{R}, dx)$ denotes the L^2-space on $\mathbb{R}$ with respect to Lebesgue measure.

1. Let $(X, \mathcal{B}, \mu)$ be a probability space, that is, X is a set, $\mathcal{B}$ is the σ-algebra of μ-measurable sets, and μ a measure so that $\mu(X) = 1$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of measurable subsets of X.

   (a) Suppose $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that
   $$\mu(x \in X : x \text{ is in infinitely many } A_n) = 0.$$

   (b) Suppose $\mu(A_n \cap A_m) = \mu(A_n)\mu(A_m)$ for all $n, m \geq 1$, and that $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) = \infty$. Show that
   $$\mu(x \in X : x \text{ is in infinitely many } A_n) = 1.$$
   You may assume that convergence in measure implies almost every convergence along a subsequence.

   (c) Show that for any $\epsilon > 0$, that for almost every $\alpha \in (0, 1)$ (with respect to Lebesgue measure), the inequality
   $$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}$$
   has only finitely many rational solutions $\frac{p}{q}$.

2. Suppose that a topological space X is compact and $f : X \to \mathbb{R}$ is a (not necessarily continuous) function. If for every $t \in \mathbb{R}$, $f^{-1}([t, \infty))$ is closed, then there is some $x_0 \in X$ so that $f(x_0) = \sup_{x \in X} f(x) < \infty$.

3. Let (X, d) be a compact metric space, and $T : X \to X$ a homeomorphism. Suppose there is a unique T-invariant Borel probability measure μ. Assume the Banach-Aloglu theorem, which states the that the unit ball in a Banach space is weak-* compact, and which you do not need to prove. Show that for any $f \in C(X)$, and any $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_X f d\mu.$$

4. Suppose $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^p(\mathbb{R}, dx)$, and set $f(x) = \int_0^x g(t)\, dt$ for $x \in \mathbb{R}$.

   (a) Show by example that f need not be differentiable at 0.

   (b) Must f have any points of differentiability? Explain.

   (c) Let $\varphi(x) = |f(x)|^q$. Show that φ is differentiable at 0 and find $\varphi'(0)$.

5. If a closed subset K of $\mathbb{R}$ is the union of half-open half-closed intervals $\{[a_\lambda, b_\lambda); \lambda \in \Lambda\}$, then it is the union of countably many of these intervals.

6. Let $(\mathbb{R}^n, \mathcal{F}, dx)$ be the Lebesgue measure space on $\mathbb{R}^n$, $1 < p < \infty$, $f \in L^1(\mathbb{R}^n, \mathcal{F}, dx)$ and $g \in L^1(\mathbb{R}^n, \mathcal{F}, dx)$. Show that $f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy$ is well-defined and that

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

   Here for $q \ge 1$, $\|h\|_q$ denotes the L^q-norm of $h \in L^q(\mathbb{R}^n, \mathcal{F}, dx)$.

7. Suppose $f \in L^1([0, 1], dx)$ and g is a bounded Lebesgue measurable periodic function on $\mathbb{R}$ with period 1 (that is, $g(x + 1) = g(x)$ for every $x \in \mathbb{R}$). Show that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \int_0^1 f(x)dx \cdot \int_0^1 g(x)dx.$$

8. Let H be a Hilbert space and $U : H \to H$ a unitary operator (that is, $U : H \to H$ is a linear operator such that $\|Uv\| = \|v\|$ for every $v \in H$). Define S_n by

$$S_n v = \frac{1}{n} \sum_{i=0}^{n-1} U^i v \quad \text{for } v \in H.$$

   (a) Prove that S_n is a bounded linear operator on H with operator norm $\|S_n\| \le 1$.

   (b) Let $I = \{v \in H : Uv = v\}$. Show for any $v \in I$ and $n \ge 0$, $S_n v = v$.

   (c) Let $B = \{Uv - v : v \in H\}$. Show that for any $w \in B$,

$$\lim_{n\to\infty} S_n w = 0,$$

   where the limit is taken in the norm topology of H.

(d) Show that $B^\perp = I$ and conclude that $H = \overline{B} \oplus I$. Here $\overline{B}$ denotes the closure of B in the Hilbert space H.

(e) Let $P : H \to I$ denote the orthogonal projection onto the invariant subspace I. Show that for any $v \in H$,

$$\lim_{n\to\infty} S_n v = Pv,$$

where the limit is taken in the norm topology of H.